The Categorical and the Hypothetical

An Inferentialist Critique of the Transmission View of Consequence

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As guest editor of the journal *Erkenntnis*, Holger Leuz planned a special issue for autumn 2008 dealing with the inferentialist approach to formal logic. Its tentative title was "Logical Inference". As this topic belonged to my core interests, I immediately agreed to submit a paper. Unfortunately, the project did not materialize — due to lack of sufficient resonance, as Holger Leuz told me later. In fact, the "boom" of publications in inferentialism and proof-theoretic semantics within general philosophical logic started slightly later, so the project was somewhat ahead of its time. By the time the project was cancelled, I had already prepared a draft. Even though it is not in polished form and its last section is lacking, it is still worth reading, in particular due to its programmatic nature. I therefore publish it here as an online resource. The topic dealt with and the theses put forward were later pursued further in several papers such as "The Categorical and the Hypothetical: A Critique of Some Fundamental Assumptions of Standard Semantics" (*Synthese* 187, 2012, 925–942, DOI: 10.1007/s11229-011-9910-z).

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The categorical and the hypothetical: An inferentialist critique of the transmission view of consequence

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Introduction

Consequence is a hypothetical concept: Claiming that B is a consequence of A asserts B with respect to the *hypothesis* A. Inference is a hypothetical notion as well: We infer something from something else which is the *hypothesis* of that inference. The distinction between consequence and inference is that consequence is a *metaphysical* notion expressing a certain relationship between sentences or propositions, whereas inference is normally understood *epistemologically* as what establishes a sentence — and therefore generates knowledge expressed by this sentence — if the hypothesis has already been established. A fundamental question concerns the explanatory order of the two concepts: Is it consequence which justifies inference and therefore comes first, or is it inference which establishes and therefore justifies a consequence statement, which would thus come second? The traditional non-inferentialist view is that consequence is the primary concept: Consequence is normally viewed as the semantical notion whereas inference proceeds by syntactically specified rules which transmit semantical validity. This is at least the view dominating in denotational semantics e.g. of the Bolzano-Tarski sort.

One should expect that an inferentialist view should reverse the order of concepts, basing the metaphysical notion on the epistemological one, and this is in fact what inferentialists with their anti-metaphysical stance claim to do. However, a closer look shows that most inferentialists proceed at best half-way in this direction, taking over fundamental prerequisites of the classical, non-inferentialist conception. In doing so, they fail to provide a conceptual framework on which an epistemological, purely inference-based approach may rest. The *fundamental misconception* of the classical view, which pertains to current inferentialism, is (1) the primacy of the categorical over the hypothetical, and (2) the transmission view of consequence. We call them the two *dogmas of standard semantics*. Their inferentialist adaptation in standard intuitionistic or proof-theoretic semantics is based on reductionist assumptions, according to which validity is ultimately reduced to atomic validity and its transmission. As an alternative, which avoids the shortcomings of standard semantics, we propose the concept of *definitional reasoning*. In Section 1 we explain what the dogmas of standard semantics are in the classical case, and in Section 2 we illustrate the way they are taken over by intuitionistic inferentialist conceptions. Section 3 relates these views to a reductionist view of reasoning. In Section 4 we lay out our alternative view of inferentialism, which takes the relation between hypotheses and consequence to be a primitive relationship. Section 5 presents some details of our conception.

1. The classical view: Truth versus consequence

According to the classical view, consequence is what preserves truth. B is a consequence of A if B is true given that A is true. In the case of *logical* consequence, this relationship is supposed to be non-accidental, i.e., independent of the particular circumstances in which B is true. This is sometimes expressed by saying that it is necessary that B is true given that A is true (such a view may perhaps be attributed to Aristotle), or that B is true in all circumstances, in which A is true. The latter view (universal truth preservation) is embodied in the modern model-theoretic view, according to which the circumstances in which a sentence is true is given by the structure in which it is evaluated. This view is sometimes traced back to Bolzano, although Bolzano's concept of "Ableitbarkeit", which corresponds to the notion of logical consequence, is more tied to the syntactic replacement of nonlogical constants within sentences rather than the evaluation of sentences in a structure[1]. The model-theoretic notion has dominated logic since Tarski's "Wahrheitsbegriff", in spite of certain deficiencies (see Etchemendy). According to the model-theoretic view, B is a consequence of A if every model of A is a model of B, i.e., if B is true in every structure, in which A is true. What is crucial for these and related notions is that they take the notion of truth (in a structure) for granted, defining consequence as preservation of this notion. As truth is a fundamental categorical notion in contradistinction to the hypothetical notion of consequence, a basic *tenet* of the classical view is:

Dogma 1. The categorical is conceptually prior to the hypothetical.

which in the model-theoretic framework means that truth in a structure is defined first and consequence as depending on it. Immediately connected with this is

Dogma 2. Consequence in defined as the transmission of the basic categorical concept from the premisses to the conclusion.

which in the model-theoretic framework means that a consequence statement is considered as valid, if truth (in a structure) is transmitted from the premisses to the

[1]Bolzano, Etchemendy.

conclusion of that statement. The primacy of the categorical over the hypothetical and, with it, the transmission view of consequence are building blocks of the classical (metaphysical) conception of consequence. This is independent of whether we consider logical consequence or some sort of "material" consequence. As remarked above, logical consequence adds a "necessary" or "universal" perspective. For our discussion in the following we just deal with consequence in an unspecified, not necessarily formal, sense. A not necessarily formal consequence statement "B follows from A" corresponds to the (not necessarily formal) truth of the implication $A \to B$.

These two dogmas are called the *dogmas of standard semantics,* as they are a background to both standard classical and standard intuitionistic or proof-theoretic semantics.

If, as an inferentialist, we adopt a more epistemological perspective, we should criticize this view. If it is inferences that justify our claim that something is true, we cannot assume that a (metaphysical) concept of truth is already at our disposal which justifies consequence. Also, we cannot rely on transmission of truth as defining the validity of what we want to establish by means of inference. To be sure, we want to be certain that a valid inference does not lead us from truth to falsity, but this should be due to intrinsic features of inference rather than due to a categorical concept which can only be generated or created by inference. One might object that the epistemological perspective of how to establish a sentence or a consequence statement is to be distinguished from the metaphysical perspective of what justifies a statement. However, this would be the metaphysicist's conception of inferentialism, not an inferentialism which takes a thoroughly epistemological standpoint.

Therefore, one should expect that the metaphysical framework is tackled by inferentialists. It is surprizing, however, that this is not the case. On the contrary, standard inferentialist approaches model their framework according to the classical picture, implicitly adapting the two dogmas. This will be illustrated by three approaches within the intuitionistic tradition, the BHK interpretation of the logical constants, Lorenzen's admissibility semantics, and proof-theoretic semantics in Dummett-Prawitz style.

2. *Consequence* in certain intuitionistic conceptions

In classical semantics, consequence statements are but a vehicle to transmit categorical truth. In the intuitionistic tradition, the concept of truth in the classical sense has been thoroughly questioned by rejecting bivalence, which is intrinsically connected to truth. Rather, the categorical concept of truth is replaced with some intuitionistic concept such as constructibility, derivability or assertibility. However, though the basic concepts are different, the framework described by the two dogmas has remained unchanged. In intuitionism, validity of consequence is still explained in terms of the transmission of a categorical concept.

2.1. BHK semantics

The so-called Brouwer-Heyting-Kolmogorov (BHK) semantics is not a uniform phenomenon, nor is the notion of hypothetical judgement[2] There is also no genuine theory of logical consequence but at best a definition of the validity of a hypothetical judgement (i.e., implication). However, the treatment of implication shows enough for us to claim that the two dogmas are subscribed. In the BHK semantics as it is usually presented (see Troelstra and van Dalen), the notion of *proof* or *construction* is taken as the basic concept in terms of which the meaning of logical constants is explained. In the following, we use the term "construction". Then the semantical explanation of implication runs as follows: A construction of "A implies B" is a procedure which transforms a construction of A into a construction of B, where "procedure" is to be understood in the sense of a constructive function, which in formal explanations such as realizability semantics is explicitly understood as a (partial) recursive function. This explanation of implication assumes that there is a categorical notion of construction which in a hypothetical statement is transmitted from the premisses to the conclusion. To be sure, this crucially differs from the classical approach in that intuitionistic "construction" is not bivalent truth in a structure, and a constructive transformation is a much stricter concept than just "preservation" of validity. However, our dogmas are not tied to a specific form of categorical concept and a specific form of transmission. In BHK semantics, we have the following intuitionistic form of the two dogmas:

Dogma 1. The categorical concept, here constructibility, is primary to the hypothetical one, here consequence or implication.

Dogma 2. Consequence is explained as transmission of the categorical concept, here by means of a constructive procedure.

This becomes explicit if we formulate the classical view of [logical] consequence of B from A as follows:

$$[\forall \mathfrak{M}](\mathfrak{M} \models A \;\Rightarrow\; \mathfrak{M} \models B) \tag{1}$$

where $\mathfrak{M}$ is a structure and $\mathfrak{M} \models A$ denotes the truth of A in $\mathfrak{M}$, and compare it to the BHK notion:

$$[\forall \mathcal{A}](\forall \mathfrak{C})(\mathfrak{C} \models_{\mathcal{A}} A \;\Rightarrow\; f(\mathfrak{C}) \models_{\mathcal{A}} B) \tag{2}$$

where $\mathfrak{C}$ denotes a construction, $\mathfrak{C} \models_{\mathcal{A}} A$ the constructibility of A by means of $\mathfrak{C}$ with respect to a set $\mathcal{A}$ of atomic constructions, and f a transformation between

[2] In particular with respect to the *ex falso quodlibet*. See van Atten (2008).

constructions. If we just consider consequence rather than logical consequence, the quantification in square brackets can be omitted. In the intuitionistic case we have, even in the case of nonlogical consequence, a universal quantifier ranging over constructions, as there the transformation is dependent on the way the premiss is constructed (with respect to $\mathcal{A}$), whereas in the model-theoretic case, there are no different ways of specifying a structure $\mathfrak{M}$. In this sense, the intuitionistic approach is intensional rather than extensional.

Viewed that way, the conceptual framework as expressed by the two dogmas remains intact. Compared to the classical case, categorical truth is replaced with a notion of categorical construction, and the "transmission view", which in the classical case is expressed by a mere metalinguistic implication, is replaced with a constructive transformation. Therefore, what we find here is a constructivization of the classical concept of consequence rather than a fundamental re-orientation. A more detailed discussion of the functional view associated with the notion of a constructive procedure or transformation will be given in Section 3.

2.2. Lorenzen's admissibility semantics

In his "Operative Logics"[3], Lorenzen developed a logical semantics which, though related to intuitionistic concepts, is more proof-theoretic in spirit. His semantics of implication is based on the notion of admissibility of rules. An implication $A \to B$ is read as a rule, and this rule is admissible in a calculus $\mathcal{K}$, if adding $A \to B$ as an inference rule to the primitive rules of of $\mathcal{K}$ does not extend the set of provable sentences, i.e., if provability of A in $\mathcal{K}$ implies provability of B in $\mathcal{K}$. However, in order to avoid the reliance on the metalinguistic implication by claiming

$$\vdash_{\mathcal{K}} A \;\Rightarrow\; \vdash_{\mathcal{K}} B,$$

Lorenzen demands that $A \to B$ be established by a constructive *elimination procedure*, which eliminates applications of $A \to B$ from proofs in $\mathcal{K} + \{A \to B\}$ (i.e., $\mathcal{K}$ extended with $A \to B$ as a primitive rule of inference). An implication is then logically valid and thus represents logical consequence, if admissibility holds in any calculus (over the language considered), in Lorenzen's terminology, if $A \to B$ is "universally admissible". Formally, this can be expressed by

$$[\forall \mathcal{K}](\forall \mathcal{D})(\forall C)(\mathcal{D} \vdash_{\mathcal{K}+\{A \to B\}} C \;\Rightarrow\; f(\mathcal{D}) \vdash_{\mathcal{K}} C) \tag{3}$$

where, as above, the quantifier in square brackets only applies if we consider universal admissibility rather than admissibility with respect to a specific $\mathcal{K}$. Here $\mathcal{D} \vdash_{\mathcal{K}} C$ expresses that $\mathcal{D}$ is a proof of C in $\mathcal{K}$, and f represents the elimination procedure which

[3] Lorenzen (1955).

transforms a proof $\mathcal{D}$ which possibly uses the rule $A \to B$ into a proof $f(\mathcal{D})$ which does not use the rule $A \to B$. Although (3) looks different from (2), it is based on a similar idea. An elimination procedure has to remove applications

$$\frac{A}{B}$$

of the rule $A \to B$ by transforming a proof of A into a proof of B without using this rule, which means that there has to be a transformation g such that

$$[\forall \mathcal{K}](\forall \mathcal{D})(\mathcal{D} \vdash_{\mathcal{K}} A \;\Rightarrow\; g(\mathcal{D}) \vdash_{\mathcal{K}} B) \tag{4}$$

The formal similarity to (2) immediately demonstrates how near this conception is to the BHK approach. The basic difference is Lorenzen's reliance on proofs and their transformations, rather than abstract notions of construction which are often used without further specification in the intuitionistic tradition. Therefore, in Lorenzen, the first dogma is expressed by the priority of provability without assumptions, and the second dogma is represented by admissibility as the transmission of provability from premisses to conclusion.

2.3. Proof-theoretic validity in the Dummett-Prawitz tradition

In several papers, Prawitz (1973, 1974, 1978, 1985, 2006) presents a definition of proof-theoretic validity which explicitly aims at providing an alternative view of consequence. It is closely related to Dummett's meaning-theoretic justification of intuitionistic logic (see Dummett 1991), and it is obvious that both approaches exerted quite some influence on one another. In the following, I rely on Prawitz's definition, which is formally more elaborated than Dummett's approach, but continue to talk of "Dummett-Prawitz" semantics.

The background to the Dummett-Prawitz approach is the theory of natural deduction, especially the theory of normalization. The central result, which motivates the philosophical definition of validity and enters there as a "fundamental assumption" (see Dummett 1991, Ch. 12), is the fact that every closed proof reduces, by certain proof transformations, to a closed proof in introduction form (i.e., using an introduction rule in the last step). Such a proof is called "canonical" by Dummett and Prawitz. A second motivation comes from semantics-like notions in normalization theory which were first proposed by Tait, Martin-Löf and Girard and first applied in demonstrations of strong normalization by Prawitz.

The Dummett-Prawitz definition of validity is relativized to a base of atomic proofs $\mathcal{S}$ and a class of proof reductions $\mathcal{J}$ and runs as follows:

- *Every atomic proof in $\mathcal{S}$ is valid.*
- *A closed canonical proof is valid, if its immediate subproofs are valid.*
- *A closed noncanonical proof is valid, if it reduces with respect to $\mathcal{J}$ to a valid closed proof.*
- *An open proof is valid, if it yields a valid closed proof, when open assumptions are substituted with valid closed canonical proofs.*

Logical validity is then validity with respect to all atomic systems $\mathcal{S}$, and also with respect to a set of proof reductions, which is independent of the atomic system chosen. For our topic most relevant is the notion of a closed canonical proof and its role in the definition of validity of open proofs. Open proofs establish consequence relations from which implications $A \to B$ can be established by $\to$-introduction in a natural-deduction style way. If $\begin{smallmatrix} A \\ \mathcal{D}' \\ B \end{smallmatrix}$ represents an open proof of B from A, and $\begin{smallmatrix} \mathcal{D} \\ A \end{smallmatrix}$ a proof of A, then the condition the last clause poses for the validity of an open proof $\begin{smallmatrix} A \\ \mathcal{D}' \\ B \end{smallmatrix}$ and thus for B being a consequence of A is that

$$[\forall \mathcal{S}](\forall \mathcal{D})(\mathcal{D} \overset{0}{\vdash}_{\mathcal{S}} A \;\Rightarrow\; \begin{matrix} \mathcal{D} \\ A \\ \mathcal{D}' \\ B \end{matrix} \overset{\mathcal{J}}{\vdash}_{\mathcal{S}} B) \tag{5}$$

if $\mathcal{D} \overset{0}{\vdash}_{\mathcal{S}} A$ expresses that $\mathcal{D}$ is a closed canonical proof of A with respect to $\mathcal{S}$, and $\mathcal{D} \overset{\mathcal{J}}{\vdash}_{\mathcal{S}} A$ expresses that $\mathcal{D}$ is, or reduces via $\mathcal{J}$ to a closed canonical proof of A. If we express the proof transformation as a function operating on proofs, we obtain

$$[\forall \mathcal{S}](\forall \mathcal{D})(\mathcal{D} \overset{0}{\vdash}_{\mathcal{S}} A \;\Rightarrow\; \mathcal{J}(\begin{matrix} \mathcal{D} \\ A \\ \mathcal{D}' \\ B \end{matrix}) \overset{0}{\vdash}_{\mathcal{S}} B) \tag{6}$$

which is very near to Lorenzen's conception with the basic difference being that now there is a derivation $\mathcal{D}'$ which mediates the transformation. However, observing that instead of $\begin{smallmatrix} A \\ \mathcal{D}' \\ B \end{smallmatrix}$ we may as well consider the single-step proof $\frac{A}{B}$ and omit the conclusion B as it is determined by the context, we obtain

$$[\forall \mathcal{S}](\forall \mathcal{D})(\mathcal{D} \overset{0}{\vdash}_{\mathcal{S}} A \;\Rightarrow\; \mathcal{J}(\mathcal{D}) \overset{0}{\vdash}_{\mathcal{S}} B) \tag{7}$$

which is formally the same as (4).[4]

[4]The analogy between Lorenzen-style and Dummett-Prawitz-style semantics is worked out in detail in Schroeder-Heister 2008b.

Dogma 1 is now instantiated as the primacy of closed canonical proofs over open proofs, and Dogma 2 as the transmission (via some proof reduction $\mathcal{J}$) of closed canonical derivability from the premisses to the conclusion.

So our result is that all three intuitionistic conceptions considered rely on a transmission view of consequence, where the categorical notion being transmitted is some notion of construction or (closed) proof, while transmission itself is viewed as a constructive function transforming categorical constructions.

3. Direct knowledge, functional knowledge, and the reductionism of standard semantics

In our remarks in the introduction and in Section 1 we criticized the two dogmas of standard semantics as being incompatible with the inferentialist approach: If inference establishes knowledge and therefore truth in the first instance, we cannot justify inference by means of the transmission of a metaphysical concept of truth. However, does this criticism apply to the variant of the two dogmas within intuitionistic approaches mentioned in the previous section? There the first dogma is given a constructive reading: Instead of metaphysical truth as the dominating categorical concept we now have a notion of constructibility (in differnt versions, viz. "construction", "proof", "canonical proof"), whereas the notion of transmission of this categorical concept from the premisses to the conclusion of an inference is now given a refined reading by interpreting it as a constructive transformation. Does this not do justice to our criticism of the metaphysical view? If we justify inference in terms of transmission of constructibility, we make sure that inference leads us to a construction, and therefore to knowledge of the conclusion, provided we have a construction, and therefore knowledge of the premisses, and this transformation is assured in terms of a constructive function. What more do we need to justify inference? From this point of view it appears that the transmission view of consequence is just what we need: We have a goal, which is expressed by a categorical concept of constructibility, and this goal is preserved by applying constructibility-preserving inferences. So the intuitionistic rendering of our two dogmas lets them appear to be just what we need to justify inferences. The main point of discussion would be which (categorical) concept of constructibility fits best to cover our knowledge, which itself leaves ample space for dispute.

The advantage of the intuitionistic view has recently been elaborated by Prawitz in several recent articles by a *regressus* argument. If we want to justify an inference step from A to B in terms of a valid consequence $A \models B$:

$$\frac{A}{B}\ A \models B$$

then the question arises what justifies to proceed from A and $A \models B$ to B, for which a higher-order principle

$$A, (A \models B) \mid\models B$$

would be needed:

$$\frac{A \quad A \models B}{B}\, A, (A \models B) \mid\models B$$

which asks for an even higher principle

$$A, (A \models B), [A, (A \models B) \mid\models B] \quad \mid\mid\models \; B$$

which would license the last inference step, etc.. This leads to an infinite regressus diagnosed by Lewis Carroll and others. The remedy according to Prawitz is that from A we obtain B not by reference to a consequence statement $A \models B$, but by reflecting on the grounds of A which gives us grounds of B by some sort of transformation. As an example he presents the left projection as a transformation which gives us a ground of A from a ground of $A \wedge B$.

Let the term t represent out knowledge of $A \wedge B$ in the form of a construction, which can be reduced to a pair of constructions of A and of B. Then we might symbolize the steps of $\wedge$ elimination by

$$\frac{t : A \wedge B}{\pi_L(t) : A} \quad \frac{t : A \wedge B}{\pi_R(t) : B}$$

In this case, the projections π_L and π_R give us grounds for the conclusion from the grounds of the premiss.

Viewed in that way, Prawitz's argument may be viewed as a philosophical defence of the two dogmas in their constructivist reading: The primary categorical concept is represented by the grounds for an assertion (in our example: t), and the transmission view is represented by the procedure which which extracts or generates grounds of the conclusion from that of the premisses, in our example π_L and π_R. Thus hypothetical knowledge is represented by means of a function, and acquiring new knowledge by means of inference means function application.

It is important to see the difference to the classical case. There what licenses us to pass from A to B is just the very fact that there is a valid consequence $A \models B$ from A to B, i.e. exactly what we want to establish by inference, whereas in the constructive case there is a function that we can apply and that delivers us the grounds for the conclusion. Applying *modus ponens* in the classical case does not supply us with grounds of the conclusion, whereas a constructive function does. Of course, we have to know what the constructive function means, i.e., in our example, that π_L and π_R are left and right projections, respectively, i.e., that the equations

$$\pi_L(\langle t_1, t_2\rangle) = t_1 \quad \pi_R(\langle t_1, t_2\rangle) = t_2$$

hold. From this point of view, hypothetical knowledge is represented by means of functions, and acquiring new knowledge by means of inference means function application. Transmission as function application appears to be the key to a proper epistemological understanding of inference.

It should be noted, however, that this conception is not so simple as it might look when only $\wedge$-elimination as presented above is considered. In the general framework, we would have to deal with assertions under assumptions, and the step from $A \wedge B$ to A would be one under a set of assumptions Γ.

$$\frac{\begin{array}{c}\Gamma \\ \vdots \\ A \wedge B\end{array}}{A}$$

Considering assumptions is essential for a proper understanding of implication, as the assertion of an implication $C \to D$ is reduced to the hypothetical assertion of D under C, i.e. the categorical knowledge of an implication $C \to D$ is reduced to the hypothetical knowledge of D under the hypothesis C. Such hypothetical knowledge is itself understood functionally, meaning that there is a function transforming a construction of C into one of D. This means that in general, an inference proceeds from a hypothetical claim to another hypothetical claim, which is to be represented as a functional operating on functions. So the example of $\wedge$-elimination would have to be represented more appropriately by:

$$\frac{t : (\Gamma \prec A \wedge B)}{(\lambda x : \Gamma)\pi_L(tx) : (\Gamma \prec A)}$$

where t is a function term representing the hypothetical knowledge of $A \wedge B$ depending on Γ (the hypothetical relationship here expressed with "$\prec$"), tx the application of t applied to knowledge of type Γ, and $(\lambda x : \Gamma)\pi_L(\bullet x)$ is the functional transmitting hypothetical knowledge of the premisses (to be inserted at the empty place $\bullet$) into hypothetical knowledge of the conclusion, i.e. the functional applied when this inference is carried out.

Another instructive example is the inference

$$\frac{A \to (B \to C)}{A \wedge B \to C}$$

again under assumptions Γ, which would have to be represented as follows:

$$\frac{t : (\Gamma \prec A \rightarrow (B \rightarrow C))}{(\lambda y : \Gamma)(\lambda x : A\wedge B)(ty(\pi_L(x))(\pi_R(x))) : (\Gamma \prec A\wedge B \rightarrow C)}$$

This shows that the transformation of knowledge (grounds) may be quite involved and needs a type-theoretic framework. Another device of handling the iteration of hypothetical knowledge would be Lorenzen's hierarchy of meta-calculi for establishing the admissibility of higher-level implications.

In the following, this sort of semantics is also called *functional semantics*, as gaining new knowledge means to apply certain functions to knowledge already gained. It is also called *constructive semantics*, as emphasis is put on the constructivity of these functions. Functional or constructive semantics is the intuitionistic form of the transmission view of consequence. The categorical concept which lies at the bottom and to which possibly iterated functions are applied are elementary constructions of proofs, to which all hypothetical knowledge is reduced.

Our argument in favour of a 'proper' inferential semantics is not that it is ill-guided, but that there can be something more powerful with constructive semantics being a limiting case. As we have seen, in constructive semantics an assertion under hypotheses is represented as functional knowledge which by means of inference is (functionally) transferred into new functional knowledge. This means that assumptions are not of equal stance with assertions, but are placeholders for constructions or closed proofs. (In a formal type-theoretic framework they are represented accordingly by variables.) Therefore, together with the primacy of the categorical over the hypothetical and the transmission view of consequence, comes what I call the *placeholder view of assumptions*: Assumptions in hypothetical judgements are but placeholders for constructions or closed proofs. One might call this the

Third dogma of standard semantics: The placeholder view of assumptions

although we prefer subsuming it under the other dogmas: Explaining the hypothetical in terms of transmission of the categorical means, at the level of hypothetical judgements, that assumptions stand for (closed) constructions which are then transmitted into a construction of the assertion. This may also be viewed as instantiating the

Fourth dogma of standard semantics: Assertion centrism

i.e., the fact that standard semantics is oriented towards forward reasoning and that hypotheses are not conceived of as backwards looking but as empty places for new assertions. Again, I subsume this under the first two dogmas, as it is a variant of them. For classical semantics it corresponds to truth-centrism.[5]

Our alternative is a view according to which the relationship between hypotheses and assertions is a basic, irreducible one. Being not assertion-centred, it does not give priority to forward reasoning. In particular, it does not subscribe to the view that all constructions can be ultimately reduced to atomic ones — a view which is intrinsically connected to the functional view of assumptions as placeholders. According to that view, a hypothetical construction of B from A transforms a construction of A into one of B, and if A is non-atomic, a construction of A is explained in terms of more elementary constructions. This means that on the transmission view of consequence, we cannot model hypothetical relationships that cannot be ultimately reduced to the atomic case.

A very simple example may serve to demonstrate this. Suppose we define a sentence A by a sentence B without B being further defined. Then we want to be able to derive A from B, even though there is no transmission of a construction of B into a construction of A, since there is no construction of B. If one objected that there is a vacuous transmission, we could suppose in addition that B is defined in terms of A. Now one might object that defining A by B and B by A is a full-fledged circularity which is to be avoided. Against that we would argue that we do have circularities in practical applications, and that a semantic theory should cover them. If we *define* A by B, then we have immediate access to the inference leading from B to A, independent of how A and B are defined otherwise. So the transmission view is not the only possible perspective. The view we propose is the *definitional* view: Laying down something by definition generates primitive inferences *even at a non-atomic level*. Viewed that way, the basic shortcoming of standard functional semantics is that there is an categorical atomic base from which knowledge emerges by (possibly iterated) transmission. Standard semantics is based on the idea that there is direct knowledge, which is atomic, and that from this direct knowledge higher level knowledge is constructed. The definitional view we advocate instead claims that there is higher-level knowledge by definition which justifies inferences. In the following section we shall explain what we have in mind.

The two dogmas of standard semantics include as a fundamental assumption that the only basic knowledge is atomic (and therefore categorical), and that all higher-level knowledge can be reduced to that in a functional way. In a sense this fundamental assumption is related to Quine's first dogma of empiricism, i.e., the idea that there is direct empirical knowledge and that everything that can be defined can ultimately be

[5]Incidentally, one might think of dualizing perspectives leading to falsity or rejection centrism, with conclusions as placeholders for refutations, a view which technically can be developed. The two basic dogmas are then fundamental again: "the categorical" is now falsity or rejection, and the "transmission view" now means that falsity or refutability is re-transmitted from conclusions to hypotheses.

defined using direct observation terms. We instead claim that definitions in general are not necessarily reductive and that, therefore, there is inferential knowledge which is not reducible (by iterated transmission) to atomic knowledge. This is related to Quine's idea that there is genuine theoretical knowledge.

So our criticism of the two dogmas of standard semantics relates to their exclusiveness. There are many phenomena that are reducible to atomic categorical constructions, in particular in the realm of formal logic, as we will later see. For example, as far as just logical constants are concerned, standard functional semantics is fine. However, there are areas that are not so well behaved, in which a proper inferentialist account is appropriate. More precisely, we propose a proper inferentialist approach as our general framework, which in certain cases reduces to the standard case in which the two dogmas are valid. Thus we object to the two dogmas as they are dogmas, i.e., demand exclusive validity.

4. Definitional reasoning

Now we outline our approach towards a non-reductionist approach to semantics, which claims to be genuinely inferential. We deliberately do not call it "logical semantics", as logical semantics, being the semantics of logical constants, is particularly well behaved and represents a field where reductionism is appropriate. In our eyes it is the fixation towards logical constants which is one of the reasons for the dominance of reductionist semantics even in the intuitionistic / constructivistic field.

Our picture of semantics as a theory of definitional reasoning is that reasoning is always related to a certain definition $\mathbb{D}$. This may be a definition of logical constants, but may apply to other constants as well. From this definition certain inference steps are generated in a way discussed in the next section.

The judgements which are generated by such inference steps are always hypothetical judgements, where these hypothetical judgements are not understood functionally. Our basic form of judgement is

$$A_1, \ldots, A_n \vdash B,$$

which is supposed to mean that B is asserted with respect to the hypotheses $A_1, \ldots, A_n$. The relationship between the assumptions or hypotheses $A_1, \ldots, A_n$ and the assertion B is considered a primitive relationship. So we consider hypothetical judgements as basic, with categorical judgements $\vdash A$ being a limiting case thereo. Thus the first dogma of standard semantics is reversed in our

First basic claim: The hypothetical is prior to the categorical.

What is generated in reasoning are genuine hypothetical statements, and not, as in constructive semantics, categorical statements with an open place for other categorical statements. Considering $A_1, \ldots, A_n \vdash B$ as primitive, implies giving up the second dogma as well, insofar as consequences are not generated from categorical constructions. However, in a sense the transmission view is upheld: Not with respect to the interpretation of hypothetical judgements $\Gamma \vdash A$, but with respect to steps from hypothetical judgements to new hypothetical judgements:

$$\frac{\Gamma_1 \vdash A_1 \quad \ldots \quad \Gamma_n \vdash A_n}{\Delta \vdash B}. \tag{1}$$

Here it is to be expected that the validity of $\Gamma \vdash B$ is generated from the validity of the $\Gamma_i \vdash A_i$. Reflection on the content of the premisses $\Gamma_i \vdash A_i$ should give us the conclusion.

However, our idea is that this is not a functional step generating a ground of $\Gamma \vdash B$ from grounds of $\Gamma_i \vdash A_i$, thus creating functional transmission at a higher level (operating on functionally understood sequents). Rather, a step such as (1) is considered a definitional step: Reflecting on the meaning of certain terms in the premisses $\Gamma_i \vdash A_i$ gives us $\Delta \vdash B$ by definition. Referring to definitional meaning avoids the regression to a justification of an even higher level: (1) is a meaning preserving step which puts a definition into action. So the second dogma of standard semantics is replaced with our

Second basic claim: Inferences are definitional steps.

This is not just another word for transforming grounds of assertions. The idea of definitional reasoning is that it proceeds always from *definiens* to *definiendum*, either in the succedent B or in the antecedent Γ of $\Gamma \vdash B$. So there is no step in which we reflect on the grounds of what we have got in order to obtain grounds for what we want to achieve. We rather introduce a *definiendum* either in the antecedent or in the succedent of a hypothetical judgement. In this sense all our inference steps are introduction inferences. The basic difference to standard proof-theoretic semantics is that we have such definitional inferences on the antecedent side as well, so we are allowed to introduce sentences both on the assertion side and on the hypothesis side. In this sense, assumptions (which are represented as antecedents of hypothetical judgements) as well as assertions (which are represented as succedents) can be inferentially manipulated. Therefore we have a genuine bi-directionality of inferences: There is no favorite side (hypothesis or assertion) which is to be manipulated in a hypothetical judgement. Formally this means giving up natural deduction as the basic model for the philosophical theory of reasoning in favour of the sequent calculus. Instead of looking at the sequent calculus solely as a meta-calculus of natural deduction (see Prawitz 1965, Appendix A), we view it as resting on basic conceptual insights which have not been exploited to far.

In our sequent-style system, $\Gamma \vdash_{\mathbb{D}} A$ expresses that the hypothetical judgement $\Gamma \vdash A$ can be proved with respect to the definition $\mathbb{D}$. Our inferential semantics will essentially motivate certain rules of the form (1) which embody reasoning with respect to the definition $\mathbb{D}$. Which rules are admissible in the end will depend on the form of the definition $\mathbb{D}$. In particular, it will depend on $\mathbb{D}$, whether a structural rule like cut

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash C}{\Gamma, \Delta \vdash C} \ (Cut)$$

is admissible. We will consider cases of $\mathbb{D}$, where cut is not admissible. For example. in the case of $\mathbb{D}$ defining a paradox, for a certain A both $\vdash_{\mathbb{D}} A$ and $A \vdash_{\mathbb{D}} \bot$ will hold, but not $\vdash_{\mathbb{D}} \bot$. This will at the same time be an example of a definition for which the transmission view is invalid. If the transmission view holds for $\mathbb{D}$, i.e., if $A \vdash_{\mathbb{D}} B$ means that $\vdash_{\mathbb{D}} A$ can be transformed into $\vdash_{\mathbb{D}} B$, then (Cut) is trivially admissible, and vice versa. Certain well-behaved definitions, but by no means all definitions, allow for the admissibility of cut. In fact, the admissibility of cut represents the validity of standard semantics and, together with it, of the transmission view of consequence. It simply depends on the given definition $\mathbb{D}$ of whether standard semantics including its two dogmas can be assumed as valid. Standard semantics is but a special case of a much wider range of phenomena open to definitional semantics.

One reason why standard semantics has dominated the tradition, including the constructivistic one, was that one was normally concerned with well-behaved cases such as logical constants, whose definition is well-founded. If we have well-founded definitions, standard semantics presents no problem whatsoever. More generally, the traditional theory of definition has been preoccupied with the well-founded case. Well-foundedness, which in explicit definitions means the eliminability of the definiendum by the definiens, has always been a demand of traditional definition theory. However, non-well-founded programs and circular and infinitely descending chains of truth definitions have widened our perspective for a more general view of definitions such as to include the non-well-founded case. As Hallnäs has pointed out, our approach corresponds to that of recursive function theory, where it is not an elementary task to decide whether a given definition of a partial recursive function defines a total function. Rather, this is a matter of (mathematical) fact. Correspondingly, we follow Halln"as in calling a definition *total* if cut is admissible, which is again nothing that can elementarily assured in all cases by the form of a definition. So totality is what makes a definition well-behaved, and which is the assumption standard semantics rests on.

This does not exclude that certain inference steps are justified by functional reasoning. For example,

$$\frac{\vdash A \wedge B}{\vdash A}$$

is easily justified that way, given the standard definition of conjunction. But these are not the general cases. If a set Γ of assumptions is present as in

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A},$$

justification by a simple admissibility consideration is not possible, as Γ may even contain $A \wedge B$. Only if we know that the definition of conjunction behaves in the standard way (which is indeed the case), we may use cut to justify ∧-elimination in the general case:

$$\frac{\Gamma \vdash A \wedge B \qquad \dfrac{A \vdash A}{A \wedge B \vdash A}}{\Gamma \vdash A} \ .$$

To repeat: An inference of the form (1) always means a reflection step: Applying a definition we introduce a defined constant into the hypotheses or into the assertion. This step is justified by reference to the definition I must know in advance. It is not a generation of grounds from grounds, i.e., a function on the knowledge of grounds, but an application of a definition, i.e., a way of extracting knowledge from a definition. In both cases new actual knowledge is generated: In the definitional case not unidirectional on the basis of grounds, but bidirectional by evaluating a definition.

References

1. Aczel, P. (1977). An introduction to inductive definitions. In: J. Barwise (ed.), *Handbook of Mathematical Logic*, North Holland: Amsterdam, 739–782.

2. Curry, Haskell B. (1958). *Outlines of a Formalist Philosophy of Mathematics*, North-Holland: Amsterdam.

3. Dummett, M. (1991). *The Logical Basis of Metaphysics*, London: Duckworth.

4. Gentzen, G. (1935). Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, **39** (1934/35), 176–210, 405–431, English translation in: M.E. Szabo (ed.), *The Collected Papers of Gerhard Gentzen*, Amsterdam: North Holland 1969, 68–131.

5. Girard, Jean-Yves. (1971). Une extension de l'interprétation de Gödel à l'analyse, et son application à l'élimination des coupures dans l'analyse et la théorie des types. In: J.E. Fenstad (Hrsg.), *Proceedings of the 2nd Scandinavian Logic Symposium (Oslo 1970)*, Amsterdam: North Holland, 63–92.

6. Hallnäs, L. (1991). Partial Inductive Definitions. *Theoretical Computer Science* **87**, 115–142.

7. Hallnäs, L. (2006). On the proof-theoretic foundation of general definition theory. In: R. Kahle & P. Schroeder-Heister (eds.), *Proof-Theoretic Semantics*. Special issue of *Synthese*, **148**(3), 589–602.

8. Hallnäs, L. & Schroeder-Heister, P. (1990/91) A proof-theoretic approach to logic programming. I. Clauses as rules. *Journal of Logic and Computation*, **1** (1990), 261–283; II. Programs as definitions, *ibid.* **1** (1991), 635–660.

9. Hermes, H. (1961). Zum Inversionsprinzip der operativen Logik. In: A. Heyting (ed.), *Constructivity in Mathematics*, Amsterdam: North-Holland, 62–68.

10. Lorenz, K. (1980). Logik, operative. Entry in: J. Ritter & K. Gründer (eds.), *Historisches Wörterbuch der Philosophie*, Vol. 5, Basel: Schwabe, 444–452.

11. Lorenz, K. (2001). Basic objectives of dialogue logic in historical perspective. *Synthese*, **127**, 255–263.

12. Lorenzen,P. (1950). Konstruktive Begründung der Mathematik. *Mathematische Zeitschrift*, **53**, 162–201.

13. Lorenzen, P. (1955). *Einführung in die operative Logik und Mathematik*. Springer: Berlin 1955, 2nd ed. Berlin 1969. (Referred to as *OL*).

14. Lorenzen, P. (1959). Über die Begriffe “Beweis” und “Definition”. In: A. Heyting (ed.), *Constructivity in Mathematics: Proceedings of the Colloquium Held at Amsterdam, 1957*, Amsterdam: North-Holland, 169–177.

15. Lorenzen, P. (1961). Ein dialogisches Konstruktivitätskriterium. In: *Infinitistic Methods: Proceedings of the Symposium on Foundations of Mathematics (Warsaw 1959)*, Oxford: Pergamon Press and Warszawa: Państwowe Wydawnictwo Naukowe, 193–200.

16. Martin-Löf, Per. (1971). Hauptsatz for the intuitionistic theory of iterated inductive definitions. In: J.E. Fenstad (Hrsg.), *Proceedings of the 2nd Scandinavian Logic Symposium (Oslo 1970)*, Amsterdam: North Holland, 179–216.

17. Moriconi, E. & Tesconi, L. (2008). On inversion principles. *History and Philosophy of Logic*), **29**, 103–113.

18. Post, E. L. (1921). Introduction to a general theory of elementary propositions. *American Journal of Mathematics*, **43**, 163–185.

19. Prawitz, D. (1965). *Natural Deduction: A Proof-Theoretical Study*, Stockholm: Almqvist & Wiksell. Reprinted Mineola, N.Y.: Dover Publ., 2006.

20. Prawitz, D. (1971). Ideas and results in proof theory. In: J.E. Fenstad (Hrsg.), *Proceedings of the 2nd Scandinavian Logic Symposium (Oslo 1970)*, Amsterdam: North Holland, 235–308.

21. Prawitz, D. (1973). Towards a Foundation of a General Proof Theory, in: P. Suppes et al. (eds.), Logic, Methodology, and Philosophy of Science IV, Amsterdam, 225–250.

22. Prawitz, D. (1974). On the idea of a general proof theory, *Synthese* **27**, 63–77.

23. Prawitz, D. (1978). Proofs and the meaning and completeness of the logical constants. In: J. Hintikka et al. (eds.), *Essays on Mathematical and Philosophical Logic*, Dordrecht: Reidel, 25–40 (revised German translation in *Conceptus* **16** (1982)).

24. Prawitz, D. (1985). Remarks on some approaches to the concept of logical consequence, *Synthese* **62**, 152–171.

25. Prawitz, D. (2006). Meaning approached via proofs. In: R. Kahle & P. Schroeder-Heister (eds.), *Proof-Theoretic Semantics*. Special issue of *Synthese*, **148**(3, 507–524.

26. Schroeder-Heister, P. (1984). A natural extension of natural deduction, *Journal of Symbolic Logic* **49**, 1284–1300.

27. Schroeder-Heister, P. (1993). Rules of definitional reflection. In: *8th Annual IEEE Symposium on Logic in Computer Science (Montreal 1993).* IEEE Computer Society Press, Los Alamitos, 222–232.

28. Schroeder-Heister, P. (2004). On the notion of *assumption* in logical systems. In: R. Bluhm & C. Nimtz (eds.), *Selected Papers Contributed to the Sections of GAP5, Fifth International Congress of the Society for Analytical Philosophy, Bielefeld, 22-26 September 2003*, Paderborn: Mentis, 27–48.

29. Schroeder-Heister, P. (2006). Validity concepts in proof-theoretic semantics. In: R. Kahle & P. Schroeder-Heister (eds.), *Proof-Theoretic Semantics*. Special issue of *Synthese*, **148**(3) (2006), 525–571.

30. Schroeder-Heister, P. (2007). Generalized definitional reflection and the inversion principle. *Logica Universalis, 1*, 355–376.

31. Schroeder-Heister, P. (2007b). Lorenzens operative Logik und moderne beweistheoretische Semantik. In: J. Mittelstraß (ed.), *Der Konstruktivismus in der Philosophie im Ausgang von Wilhelm Kamlah und Paul Lorenzen* (Beiträge zum Symposium in Konstanz, September 2005). Paderborn: Mentis, 167–196.

32. Schroeder-Heister, P. (2008a). Proof-theoretic vs. model-theoretic consequence. In: *The Logica 2007 Yearbook*. Prague: Filosofia, 187–200.

33. Schroeder-Heister, P. (2008b). Lorenzen's operative justification of intuitionistic logic. In: M. van Atten, P. Boldini, M. Bourdeau & G. Heinzmann (eds.), One Hundred Years of Intuitionism (1907-2007), Basel: Birkhäuser 2008.

34. Schroeder-Heister, P. & Contu, P. (2005). Folgerung. In: W. Spohn, P. Schroeder-Heister & E.J. Olsson (eds.), *Logik in der Philosophie*, Heidelberg: Synchron Wissenschaftsverlag, 247–276.

35. Tait, William W. 1967. Intensional interpretations of functionals of finite type I. *Journal of Symbolic Logic* **32**. 198–212.