# The Yang–Mills Vacuum Wave Functional in Coulomb Gauge

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*Amore et pietate parentibus meis*

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## Zusammenfassung

Yang-Mills-Theorien sind die Grundlage des heutigen Standardmodells der Elementarteilchenphysik. Neben Methoden, die auf eine Diskretisierung der Raum-Zeit basieren (Gittereichtheorie), sind auch analytische Zugänge möglich, sowohl im Lagrange- als auch im Hamilton-Formalismus. Diese Dissertation behandelt die Hamilton'sche Formulierung von Yang-Mills-Theorien in Coulomb-Eichung.

Die Dissertation ist in kumulativer Form verfasst. Nach einer Einführung in die allgemeine Formulierung von Yang-Mills-Theorien wird der Hamiltonoperator in Coulomb-Eichung hergeleitet.

In Kap. 1 wird die Faddeev-Popov-Determinante mittels einer Heat-Kernel-Entwicklung untersucht.

In Kap. 2 und 3 wird das Hochenergie-Verhalten der Theorie untersucht. Dafür werden störungstheoretische Methoden verwendet und die Ergebnisse werden mit Resultaten aus funktionalen Methoden in Coulomb- und Landau-Eichung verglichen.

Kap. 4 ist dem Variationszugang gewidmet. Es werden Dyson-Schwinger-Techniken verwendet, um über die bisher betrachteten Gauß'schen Ansätze für das Vakuumfunktional hinaus zu gehen. Gleichungen für Variationsparameter höherer Ordnung werden hergeleitet und die Effekte der neu auftretenden Terme abgeschätzt.

Kap. 5 beinhaltet eine Anwendung der früher hergeleiteten nichtstörungstheoretischen Propagatoren, nämlich die Berechnung der topologischen Suszeptibilität, welche mit der Masse des $\eta'$-Mesons verknüpft ist.

Schließlich wird eine kurze Einführung in die störungstheoretische Behandlung von dynamischen Fermion-Feldern gegeben.

## Abstract

Yang–Mills theories are the building blocks of today's Standard Model of elementary particle physics. Besides methods based on a discretization of space-time (lattice gauge theory), also analytic methods are feasible, either in the Lagrangian or in the Hamiltonian formulation of the theory. This thesis focuses on the Hamiltonian approach to Yang–Mills theories in Coulomb gauge.

The thesis is presented in cumulative form. After an introduction into the general formulation of Yang–Mills theories, the Hamilton operator in Coulomb gauge is derived.

Chap. 1 deals with the heat-kernel expansion of the Faddeev–Popov determinant.

In Chapters 2 and 3, the high-energy behaviour of the theory is investigated. To this purpose, perturbative methods are applied, and the results are compared with the ones stemming from functional methods in Coulomb and Landau gauge.

Chap. 4 is devoted to the variational approach. Variational ansatzes going beyond the Gaussian form for the vacuum wave functional are considered and treated using Dyson–Schwinger techniques. Equations for the higher-order variational kernels are derived and their effects are estimated.

Chap. 5 presents an application of the previously obtained propagators, namely the evaluation of the topological susceptibility, which is related to the mass of the $\eta'$ meson.

Finally, a short overview of the perturbative treatment of dynamical fermion fields is presented.

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Ταράσσει τοὺς ἀνθρώπους οὐ τὰ πράγματα, ἀλλὰ τὰ περὶ τῶν πραγμάτων δόγματα.

Epictetus

*('Not things, but opinions about things, trouble folk.')*

Incidis in Scyllam
cupiens vitare Charybdin.

*(Gautier de Châtillon)*

# Hamiltonian approach to Yang–Mills theories in Coulomb gauge

## I.1 Introduction

The Standard Model (SM) of particle physics is the current framework to describe the fundamental interactions between elementary particles with the exception of gravity. The underlying tool is Quantum Field Theory (QFT), which applies to fields the quantization methods originally developed for particles.

In its present form, the SM describes the strong, weak, and electromagnetic interactions via gauge theories. The principle of gauge invariance was originally introduced by Weyl [1] in an attempt to find a unified formulation of classical electrodynamics and general relativity. Later on, it was reintroduced in 1954 by Yang and Mills [2] with the less ambitious purpose of describing the interaction among nucleons, making out of the global isospin symmetry a local one. Despite failing this original goal, for it predicted the existence of unobserved massless particles, the theory was not entirely forgotten. Descriptions of weak and strong interactions based on gauge theories originated in the 1960s and early 1970s [3,4]. Issues about renormalization and symmetry breaking were (partially) solved later on [5].

Yang–Mills theories are still far from being well understood. They are a remarkable piece of theoretical complexity, for they display a wide spectrum of properties typical for quantum field theories, ranging from quantum anomalies to symmetry breaking. A renormalization proof in every gauge is yet to be achieved, and the long-range behaviour is still being investigated, along with symmetry breaking patterns.

Quantum Chromodynamics (QCD) is the Yang–Mills theory based on the gauge group $SU(3)$. It describes the interaction between particles, quarks and gluons, which carry a so-called 'colour' charge. Although deep inelastic scattering experiments validated QCD as the correct theory at high energies (where the smallness of the coupling allows perturbative methods to be applied), its low-energy or infrared (IR) behaviour is still not well understood. Comprehending the IR sector of QCD is one of the most challenging problems of today's theoretical particle physics. The fact that the coupling constant becomes large in the IR region prevents one from applying perturbative methods.

The foremost distinctive feature of QCD is the so-called *colour confinement*: although the quark fields carry a colour charge, the observed physical hadrons do not. This is a genuinely non-perturbative aspect of the theory, which requires adequate tools to deal with. The only *ab-initio* approach to QCD is *lattice gauge theory* [6], based on the discretization of space-time. Lattice calculations call for high computational costs, in particular when dealing with dynamical fermions. Furthermore, a complete understanding of the theory

cannot come from a single approach, and continuum methods must be considered too.

In the last few years there has been a considerable effort in the continuum formulation of Yang–Mills theories, which in turn has led to some progress, mostly within the framework of Dyson–Schwinger equations in the Landau gauge [7]. More recently, Coulomb gauge functional studies have been performed at the perturbative level [8] and in the heavy-quark sector [9]. Furthermore, the approach based on the Hamiltonian formulation of the theory in Coulomb gauge has opened up new lines of research [10–12].

This chapter is an introduction to the Hamiltonian formulation of Yang–Mills theory. Starting with considerations on the gauge invariance of the free fermion theory, the Yang–Mills Lagrangian will be derived, together with the corresponding Hamiltonian. The issue of gauge invariance in the classical theory will be discussed, and the theory will be then canonically quantized. After a discussion about residual gauge invariance and topological aspects of the theory, the Yang–Mills Hamilton operator in Coulomb gauge will be given, which is the starting point for the investigations comprised in this thesis.

## I.2 Classical formulation of Yang–Mills theory

### I.2.1 From gauge invariance to the classical Hamiltonian

It is well known that in electrodynamics the scalar and the vector potentials $A_0$ and $\mathbf{A}$ can be introduced to describe the electric and magnetic fields, namely[1]

$$\mathbf{E} = -\boldsymbol{\nabla} A_0 - \frac{\partial \mathbf{A}}{\partial t}\,, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{I.1}$$

which automatically satisfy two of Maxwell's equations

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0. \tag{I.2}$$

The remaining two Maxwell equations (Gauss's and Ampère's law) can then be written as equations for the potentials. However, it is also known that these potentials are not uniquely defined: since the magnetic field $\mathbf{B}$ is defined as the curl of the vector potential, adding to $\mathbf{A}$ the gradient of an arbitrary space-time dependent scalar function $\Lambda(\mathbf{x}, t)$ does not change the magnetic field,

$$\boldsymbol{\nabla} \times (\mathbf{A}(\mathbf{x}, t) + \boldsymbol{\nabla}\Lambda(\mathbf{x}, t)) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x}, t). \tag{I.3}$$

The electric field $\mathbf{E}$ remains unchanged, as long as the time derivative of the same gauge function is subtracted from the scalar potential,

$$A_0(\mathbf{x}, t) \to A_0(\mathbf{x}, t) - \frac{\partial \Lambda(\mathbf{x}, t)}{\partial t}\,. \tag{I.4}$$

Upon formulating quantum mechanics, it was soon realized that wave functions have to transform too, in order to maintain gauge independence of the physical quantities. This is most easily seen in the covariant form of the Lagrangian density of Quantum Electrodynamics (QED)

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\big(\mathrm{i}\partial\!\!\!/ - e A\!\!\!/ - m\big)\psi, \tag{I.5}$$

[1] Natural units $\hbar = c = 1$ and the Heaviside-Lorentz system are used.

where $\psi$ is the fermion field, and $e$ the electron charge. In Eq. (I.5), the slashed notation stands for the contraction with the Dirac matrices $\not{\partial} = \gamma^\mu \partial_\mu$, and $F_{\mu\nu}$ is the abelian field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{I.6}$$

Under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{I.7}$$

the field strength tensor Eq. (I.6), and hence the first term of the QED Lagrangian Eq. (I.5), remains unchanged. However, the fermion-photon mixing term $\bar{\psi} \not{A} \psi$ stemming from the minimal coupling prescription yields an additional term. In order to preserve the invariance of the Lagrangian Eq. (I.5), it must be assumed that the fermion field transforms according to

$$\psi(x) \to \mathrm{e}^{-\frac{\mathrm{i}}{e}\Lambda(x)} \psi(x). \tag{I.8}$$

The transformations of the type described in Eq. (I.8) (unimodular complex numbers) form the unitary group $U(1)$.

In order to generalize these ideas to other groups, it is common to start from the free Dirac Lagrangian

$$\mathcal{L}_{\mathrm{D}} = \bar{\psi}(\mathrm{i}\not{\partial} - m)\psi. \tag{I.9}$$

It is then assumed that the fermion field $\psi$ carries some kind of internal quantum number, and transforms in a representation $R$ of a group

$$\psi^m(x) \to U^{mn}\, \psi^n(x), \tag{I.10}$$

where the indices $m$, $n$ run from 1 to the dimension $\dim(R)$ of the representation $R$. For a unitary (or orthogonal) group, the Lagrangian Eq. (I.9) remains invariant for any constant $U$. On the other hand, whilst the mass term in Eq. (I.9) is invariant under a *local* transformation of the type

$$\psi^m(x) \to U^{mn}(x)\, \psi^n(x), \tag{I.11}$$

the derivative acting on the transformed field brings in an additional term. To preserve the invariance of the Dirac Lagrangian Eq. (I.9), the standard derivative $\partial_\mu$ appearing in Eq. (I.9) must be replaced by a *covariant derivative* $D_\mu$, which should transform as

$$D_\mu \xrightarrow{U} U(x)\, D_\mu U^\dagger(x). \tag{I.12}$$

Inspired by the QED case, a form dictated by a minimal coupling prescription is usually chosen for the covariant derivative,

$$D_\mu = \partial_\mu + \mathrm{i}\, g A_\mu(x), \tag{I.13}$$

where $A_\mu$ is a (matrix valued) gauge field, and $g$ is the *coupling constant*. Parametrizing the transformation $U(x)$ as[2]

$$U(x) = \exp\bigl[\mathrm{i}\, t_a^{(R)} \varphi^a(x)\bigr], \tag{I.14}$$

[2] In the following, the focus is on the special unitary group $SU(N_{\mathrm{c}})$. The $t_a^{(R)}$'s are Hermitian matrices building an irreducible representation $R$ of the underlying $\mathfrak{su}(N_{\mathrm{c}})$ algebra, and satisfying the commutation rules

$$\bigl[t_a^{(R)}, t_b^{(R)}\bigr] = \mathrm{i}\, f^{abc}\, t_c^{(R)},$$

where the $f^{abc}$ are the totally antisymmetric structure constants.

Eq. (I.11) reads in infinitesimal form

$$\psi^m(x) \to \psi^m(x) + \mathrm{i}\,\varphi^a(x)\,(t_a^{(R)})^m{}_n\,\psi^n(x), \tag{I.15}$$

which implies that the gauge field itself must be an element of the algebra, i.e.

$$A_\mu(x) = A_\mu^a(x)\,t_a^{(R)}. \tag{I.16}$$

From the transformation property Eq. (I.12) written explicitly for Eq. (I.13)

$$\partial_\mu + \mathrm{i}\,g A_\mu^U = U\big(\partial_\mu + \mathrm{i}\,g A_\mu\big)U^\dagger \tag{I.17}$$

the transformation property for the gauge field can be derived

$$A_\mu^U = U A_\mu U^\dagger - \frac{\mathrm{i}}{g}\,U\big(\partial_\mu U^\dagger\big) = -\frac{\mathrm{i}}{g}\,U(D_\mu U^\dagger). \tag{I.18}$$

In order to write down a kinetic term for the new field, a field strength tensor $F_{\mu\nu}$ can be defined in analogy to the QED case

$$F_{\mu\nu} := -\frac{\mathrm{i}}{g}\,\big[D_\mu, D_\nu\big] = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}\,g\big[A_\mu, A_\nu\big]. \tag{I.19}$$

Being defined as the commutator of two covariant derivatives, the field strength tensor $F_{\mu\nu}$ transforms covariantly like $D_\mu$,

$$F_{\mu\nu} \xrightarrow{U} U F_{\mu\nu}\,U^\dagger. \tag{I.20}$$

The kinetic term is defined in analogy with to QED, and the gauge invariant Lagrangian can be eventually written down as

$$\mathcal{L}_{\mathrm{YM+D}} = -\frac{1}{4T_R}\;\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}] + \bar\psi(\mathrm{i}\not{D} - m)\psi, \tag{I.21}$$

where $T_R$ is the trace invariant of the representation $R$, defined by

$$\mathrm{tr}\big(t_a^{(R)} t_b^{(R)}\big) = T_R\,\delta^{ab}. \tag{I.22}$$

($T_F = \frac{1}{2}$ for the fundamental representation of the $\mathfrak{su}(N_\mathrm{c})$ algebra.)

Besides the "historical" derivation of the Yang–Mills Lagrangian presented here, a more formal approach exploits the close connection to gravity, on which modern gauge-gravity unifying speculations rely. The underlying idea is that, since fields at different space-time points transform differently, the normal derivative cannot be a sensitive measure of the change rate of a field. Comparing Eq. (I.13) rewritten with explicit indices

$$D_\mu\psi^m(x) = \partial_\mu\psi^m(x) + \mathrm{i}\,g A_\mu^a(x)\big(t_a^{(R)}\big)^m{}_n\,\psi^n(x) \tag{I.23}$$

with the covariant derivative of a vector field $V^\nu$ on a Riemannian manifold

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho, \tag{I.24}$$

the components of the gauge field can be interpreted as the coefficients of an affine connection

$$\Gamma^m_{\mu n} = \mathrm{i}\,g A_\mu^a(x)\big(t_a^{(R)}\big)^m{}_n\,. \tag{I.25}$$

It should be clear that Eq. (I.25) has to be interpreted with care, since all indices of the connection coefficients in Eq. (I.24) run over the number of space-time dimensions, while in Eq. (I.25) there are one space-time and two internal symmetry indices.

Both in the original approach (i.e. guaranteeing gauge invariance of the Dirac Lagrangian) as well as in the geometrical one (comparing Dirac fields at different points), matter fields are the basis for the introduction of gauge fields. Furthermore, in the case of the $U(1)$ group (QED) the theory without matter fields would be quite unimpressive, although certain subtleties related to gauge invariance do arise also in this case. However, it is for non-abelian groups that the theory becomes highly non-trivial. Due to the self-coupling of the Yang–Mills fields, stemming from the commutator term in Eq. (I.19), the gauge sector itself becomes compelling. From this time onwards, pure Yang–Mills fields without dynamical matter will be considered, i.e. the Lagrangian is restricted to

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}\, F^a_{\mu\nu}\, F^{a,\mu\nu} \; - g\, j^{a,\mu} A^a_\mu \,. \tag{I.26}$$

When dealing with the Yang–Mills Lagrangian with external currents $j^a_\mu$, gauge invariance seems to be lost. If gauge invariance in the presence of external currents has to be preserved, the Lagrangian Eq. (I.26) should be modified. This subtlety will henceforth be ignored, and the pure Yang–Mills Lagrangian with external currents as given in Eq. (I.26) should be understood as a condensed notation for the full interacting theory defined by Eq. (I.21).

To make the connection to electrodynamics apparent, and since both the Hamiltonian approach and the Coulomb gauge (which will be used later on) explicitly break Lorentz invariance, the relativistic notation will be abandoned. The chromoelectric field is introduced as

$$E^a_i = -\partial_0 A^a_i - \hat{D}^{ab}_i A^b_0, \tag{I.27}$$

where the spatial part of the covariant derivative in the adjoint representation of the colour group is

$$\hat{D}^{ab}_i \equiv \delta^{ab}\partial_i + g\hat{A}^{ab}_i, \quad \hat{A}^{ab}_i \equiv f^{acb} A^c_i. \tag{I.28}$$

Here and in the following, Lorentz subscripts refer to vector indices, so that Eqs. (I.27) and (I.28) read

$$\mathbf{E}^a = -\partial_0 \mathbf{A}^a - \mathbf{D}^{ab} A^b_0, \tag{I.27$'$}$$

$$\mathbf{D}^{ab} = \delta^{ab}\boldsymbol{\nabla} + g\mathbf{A}^{ab}. \tag{I.28$'$}$$

The Yang–Mills Lagrangian Eq. (I.26) takes the form

$$\mathcal{L}_{\text{YM}} = \frac{1}{2}\, E^a_i E^a_i - \frac{1}{4}\, F^a_{ij} F^a_{ij} - g\rho^a_{\text{ext}} A^a_0 + j^a_i A^a_i, \tag{I.29}$$

where

$$F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + g\, f^{abc} A^b_i\, A^c_j \tag{I.30}$$

is the spatial part of the field strength tensor. In three spatial dimensions the chromomagnetic field can be introduced by

$$B^a_i = \frac{1}{2}\, \varepsilon_{ijk} F^a_{jk}\,, \qquad \mathbf{B}^a = \boldsymbol{\nabla} \times \mathbf{A}^a + \frac{g}{2}\, f^{abc}\, \mathbf{A}^b \times \mathbf{A}^c, \tag{I.31a}$$

which can be inverted to give the field strength tensor in terms of the chromomagnetic field

$$F_{ij}^a = \varepsilon_{ijk} B_k^a \,. \tag{I.31b}$$

With Eq. (I.31b), the second term in the Yang–Mills Lagrangian Eq. (I.29) can be written as

$$\frac{1}{4}\, F_{ij}^a F_{ij}^a = \frac{1}{2}\, B_i^a B_i^a \qquad (d=3). \tag{I.32}$$

In the following, the number $d$ of spatial dimensions will be left unfixed: nevertheless, the second term of Eq. (I.29) will be referred to as the 'magnetic term'.

Varying the action

$$\mathcal{S}_{\rm YM} = \int \mathrm{d}^{d+1}x\, \mathcal{L}_{\rm YM} \tag{I.33}$$

the Euler–Lagrange equations can be derived in the usual way. The equation following the variation with respect to the scalar potential $A_0^a$ is Gauss's law

$$\hat{D}_i^{ab} E_i^b = g\rho_{\rm ext}^a \,, \tag{I.34}$$

while the variation with respect to the vector potential $A_i^a$ gives the generalization of Ampère's law

$$\hat{D}_j^{ab} F_{ij}^b = g\, j^a + \hat{D}_0^{ab} E_i^b, \tag{I.35}$$

where the temporal component of the covariant derivative in the adjoint representation reads

$$\hat{D}_0^{ab} \equiv \delta^{ab}\partial_0 - g\hat{A}_0^{ab}. \tag{I.36}$$

[Notice the sign difference with Eq. (I.28).] In three spatial dimensions, Eq. (I.35) can be written in vector form as

$$\mathbf{D}^{ab} \times \mathbf{B}^b = g\,\mathbf{j}^a + \hat{D}_0^{ab}\mathbf{E}^b. \tag{I.35$'$}$$

The conjugated momenta can be derived from the Lagrangian density Eq. (I.29) in the usual way. Time derivatives occur only in the electric field, and a simple calculation shows

$$\Pi_0^a = \frac{\delta \mathcal{S}_{\rm YM}}{\delta(\partial_0 A_0^a)} = 0, \tag{I.37a}$$

$$\Pi_i^a = \frac{\delta \mathcal{S}_{\rm YM}}{\delta(\partial_0 A_i^a)} = E_j^b \frac{\partial E_j^b}{\delta(\partial_0 A_i^a)} = -E_i^a. \tag{I.37b}$$

The classical Hamiltonian (density) is obtained by Legendre transformation

$$\mathcal{H} = \frac{1}{2}\,\Pi_i^a\,\Pi_i^a + \frac{1}{4}\,F_{ij}^a\,F_{ij}^a + A_0^a\big(g\rho_{\rm ext}^a + \hat{D}_i^{ab}\Pi_i^b\big), \tag{I.38}$$

after having performed an integration by parts and discarded the surface terms. The Hamilton equations of motion follow from Eq. (I.38) in the standard way. The equations for $\dot{A}_i^a$ and $\dot{\Pi}_i^a$ result, as expected, in Eqs. (I.27) and (I.35), with the electric field $E_i^a$ replaced by $-\Pi_i^a$, in compliance with Eq. (I.37b). Furthermore, the equation for the (vanishing) momentum $\Pi_0^a$

$$0 = -\dot{\Pi}_0^a = \frac{\partial \mathcal{H}}{\partial A_0^a} = g\rho_{\rm ext}^a + \hat{D}_i^{ab}\Pi_i^b \tag{I.39}$$

is nothing but Gauss's law. Notably, since there is no momentum conjugated to the scalar potential, the equation for the time derivative of the latter reads

$$\dot{A}_0^a = \frac{\partial \mathcal{H}}{\partial \Pi_0^a} = 0, \tag{I.40}$$

which implies that the scalar potential is not a dynamical variable. Indeed, due to the antisymmetry of the field strength tensor Eq. (I.19), no time derivative of the scalar potential $A_0^a$ enters the Lagrangian (or the Hamiltonian). The $A_0^a$ field plays the role of a Lagrange multiplier, and Gauss's law can be seen as a constraint rather than a dynamical equation of motion.

### I.2.2 On the simultaneous choice of Coulomb and Weyl gauge

The issue of gauge freedom in the classical theory, in particular the simultaneous choice of the temporal (or Weyl) and the Coulomb gauge, will now be briefly discussed. For the sake of simplicity, the discussion will be restricted to the Maxwell theory with static external sources. The Lagrangian density as a function of the scalar potential $A_0$ and the vector potential $\mathbf{A}$ with given external charges $\rho$ reads

$$\mathcal{L} = \frac{1}{2}(\boldsymbol{\nabla} A_0 + \dot{\mathbf{A}})^2 - \frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{A})^2 - \rho A_0. \tag{I.41}$$

The action defined by the Lagrangian Eq. (I.41) is invariant under the following local gauge transformations

$$A_0(\mathbf{x}, t) \to A_0(\mathbf{x}, t) - \dot{\Lambda}(\mathbf{x}, t), \quad \mathbf{A}(\mathbf{x}, t) \to \mathbf{A}(\mathbf{x}, t) + \boldsymbol{\nabla}\Lambda(\mathbf{x}, t), \tag{I.42}$$

provided that the external sources satisfy $\dot{\rho} = 0$. The Euler–Lagrange equations derived from the given Lagrangian are

$$\begin{cases} -\nabla^2 A_0 - \boldsymbol{\nabla} \cdot \dot{\mathbf{A}} = \rho \\ \partial^2 \mathbf{A} + \boldsymbol{\nabla}(\dot{A}_0 + \boldsymbol{\nabla} \cdot \mathbf{A}) = 0. \end{cases} \tag{I.43}$$

These equations are form invariant under the gauge transformations Eq. (I.42).

To fix the Coulomb gauge, the gauge function

$$\Lambda(\mathbf{x}, t) = \frac{1}{-\nabla^2}\, \boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{x}, t), \tag{I.44}$$

can be used, and the transformed vector potential $\mathbf{A}^\perp$ is purely transverse, $\boldsymbol{\nabla} \cdot \mathbf{A}^\perp = 0$. The Lagrangian reads in terms of the transformed fields

$$\mathcal{L}_{\text{Coul}} = \frac{1}{2}(\boldsymbol{\nabla} A_0 + \dot{\mathbf{A}}^\perp)^2 - \frac{1}{2}(\boldsymbol{\nabla} \times \mathbf{A}^\perp)^2 - \rho A_0, \tag{I.45}$$

and the resulting equations of motion are

$$\begin{cases} -\nabla^2 A_0(\mathbf{x}, t) = \rho(\mathbf{x}) \\ \partial^2 \mathbf{A}^\perp(\mathbf{x}, t) + \boldsymbol{\nabla}\dot{\phi}(\mathbf{x}, t) = 0. \end{cases} \tag{I.46}$$

The Coulomb gauge Lagrangian Eq. (I.45) has still a residual gauge freedom: since the gauge transformation of the vector potential consists always in adding a longitudinal term [see Eq. (I.42)], if the vector potential has to remain transverse, the "allowed" transformations cannot be space dependent.[3] With a gauge function $\lambda(t)$ of time only, the scalar potential transforms as

$$A_0(\mathbf{x},t) \to A_0(\mathbf{x},t) - \frac{\mathrm{d}\lambda(t)}{\mathrm{d}t}\,, \tag{I.47}$$

while the vector potential $\mathbf{A}$ remains unaltered. It is possible to get a vanishing scalar potential at a single point in space, but the global condition $A_0(\mathbf{x},t) = 0$ cannot be enforced. Indeed, the scalar potential is constrained by Gauss's law

$$A_0(\mathbf{x},t) = \frac{1}{-\boldsymbol{\nabla}^2}\,\rho, \tag{I.48}$$

which completely specifies the scalar potential in terms of the charge distribution. Then "inserting" Gauss's law in the Lagrangian Eq. (I.45), it is possible to obtain an effective Lagrangian which takes care of the constraint Eq. (I.48)

$$\begin{aligned}\mathcal{L}_{\text{Coul+Gauss}} &= \frac{1}{2}\big(\dot{\mathbf{A}}^{\perp} + \boldsymbol{\nabla}\frac{1}{-\boldsymbol{\nabla}^2}\,\rho\big)^2 - \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{A}^{\perp})^2 - \rho\,\frac{1}{-\boldsymbol{\nabla}^2}\,\rho \\ &= \frac{1}{2}(\dot{\mathbf{A}}^{\perp})^2 - \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{A}^{\perp})^2 - \frac{1}{2}\rho\,\frac{1}{-\boldsymbol{\nabla}^2}\,\rho.\end{aligned} \tag{I.49}$$

(It is always implicitly assumed that this density is integrated over and surface terms can be discarded.)

Alternatively, the temporal gauge can be fixed by means of the gauge function

$$\Lambda(\mathbf{x},t) = \int_0^t \mathrm{d}t'\; A_0(\mathbf{x},t'). \tag{I.50}$$

In the temporal gauge, the Lagrangian reads

$$\mathcal{L}_{\text{Weyl}} = \frac{1}{2}\dot{\mathbf{A}}^{\,2} - \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{A})^2, \tag{I.51}$$

and only one equation of motion is recovered

$$\partial^2\mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) = 0, \tag{I.52}$$

while Gauss's law formally disappears. Time-independent gauge transformations can still be performed,

$$\mathbf{A}(\mathbf{x},t) \to \mathbf{A}(\mathbf{x},t) + \boldsymbol{\nabla}\lambda(\mathbf{x}), \tag{I.53}$$

which leave the gauge condition $A_0 = 0$ unaltered. It could then be possible to perform a gauge transformation which brings the vector potential in the Coulomb gauge; however, this can be done only at a fixed time. The gauge function implementing the Coulomb gauge is different at any time slice: again, the Weyl and Coulomb gauges cannot be enforced

[3] One could exploit harmonic functions, but imposing suitable regularity and boundary conditions, these can be ignored.

simultaneously at all times. However, being bold and imposing the would-be Gauss law by setting $A_0 = 0$ in Eq. (I.43) gives

$$-\boldsymbol{\nabla}\cdot\dot{\mathbf{A}} = \rho \quad \Rightarrow \quad \dot{\mathbf{A}} = \dot{\mathbf{A}}^{\perp} + \boldsymbol{\nabla}\frac{1}{-\nabla^2}\,\rho. \tag{I.54}$$

Inserting this into the Weyl Lagrangian (I.51), the constrained Weyl Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{Weyl+Gauss}} &= \frac{1}{2}\big(\dot{\mathbf{A}}^{\perp} + \boldsymbol{\nabla}\frac{1}{-\nabla^2}\,\rho\big)^2 - \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{A}^{\perp})^2 \\ &= \frac{1}{2}\,(\dot{\mathbf{A}}^{\perp})^2 + \frac{1}{2}\,\rho\,\frac{1}{-\nabla^2}\,\rho - \frac{1}{2}(\boldsymbol{\nabla}\times\mathbf{A}^{\perp})^2,\end{aligned} \tag{I.55}$$

is obtained, where the surface terms arising from the integration by parts have been discarded. It is then apparent that $\mathcal{L}_{\text{Coul+Gauss}} = \mathcal{L}_{\text{Weyl+Gauss}}$. The implementation of the Coulomb gauge with subsequent inclusion of Gauss's law constraint yields the same effective Lagrangian as the Weyl gauge fixing procedure followed by implementation of the would-be Gauss law.

In both cases, the result is an effective Lagrangian which depends only on the transverse degrees of freedom $\mathbf{A}^{\perp}$ and the external sources, but no two consecutive gauge transformations have been performed. Only one gauge has been fixed by means of a suitable transformation, and then the constraint of Gauss's law has been used to eliminate the residual superfluous degrees of freedom.

Both approaches are completely equivalent at the classical level. At the quantum level, however, only the second one is feasible because of the well-known problem of the vanishing momentum conjugate to the scalar potential in the canonical formalism.

## I.3 Canonical quantization and the $\theta$-vacuum

### I.3.1 Hamilton operator in the temporal gauge and Schrödinger picture

The starting point for the canonical quantization of Yang–Mills theories is the classical Hamiltonian Eq. (I.38), which should be promoted to an operator by imposing canonical commutation relations on the conjugate variables $A_i^a$ and $\Pi_i^a = -E_i^a$. It has been remarked in the previous section that the canonical momentum conjugate to $A_0^a$ vanishes. This is not a real problem for the classical theory (Sec. I.2.2), but troubles arise when canonical commutation relations are imposed. This difficulty can be circumvented by making use of the gauge freedom of the Lagrangian to eliminate the scalar potential, i.e. choosing the Weyl gauge $A_0^a = 0$.[4] The canonical, equal-time commutation relations are then imposed

$$\big[A_i^a(\mathbf{x}), A_j^b(\mathbf{y})\big] = 0 = \big[\Pi_i^a(\mathbf{x}), \Pi_j^b(\mathbf{y})\big], \tag{I.56a}$$

$$\big[A_i^a(\mathbf{x}), \Pi_j^b(\mathbf{y})\big] = \mathrm{i}\,\delta^{ab}\,\delta_{ij}\,\delta(\mathbf{x}-\mathbf{y}), \tag{I.56b}$$

[4] The generalization of the gauge transformation Eq. (I.50) which realizes the transition to the Weyl gauge in the non-abelian case is the solution of the equation $D_0 U^{\dagger} = 0$, which is easily solved to find

$$U(\mathbf{x},t) = \mathsf{T}\exp\left\{-\mathrm{i}g\int_0^t \mathrm{d}\tau\, A_0(\mathbf{x},\tau)\right\},$$

with $\mathsf{T}$ being the time ordering symbol.

and the familiar Hamilton operator for the gauge field is recovered,

$$H = \int \mathrm{d}^d x \left[ \frac{1}{2}\, \Pi_i^a(\mathbf{x})\, \Pi_i^a(\mathbf{x}) + \frac{1}{4}\, F_{ij}^a(\mathbf{x})\, F_{ij}^a(\mathbf{x}) \right]. \tag{I.57}$$

As in quantum mechanics, in the Schrödinger picture [13] one works with a representation of the more abstract state vectors. In this work, the "coordinate" representation will be used, i.e. wave functionals which are related to the state vectors $|\,\rangle$ by

$$\psi[A] = \langle A|\,\rangle, \tag{I.58}$$

where $|A\rangle$ denotes the eigenstate of the field *operator* $\hat{\mathbf{A}}$

$$\hat{A}_i^a(\mathbf{x})\,|A\rangle = A_i^a(\mathbf{x})\,|A\rangle. \tag{I.59}$$

The commutation relations (I.56) are then satisfied by realizing the field operator multiplicatively

$$\langle A|\,\hat{A}_i^a(\mathbf{x})\,|\,\rangle = A_i^a(\mathbf{x})\,\psi[A], \tag{I.60a}$$

and the momentum operator as a functional derivative

$$\langle A|\,\hat{\Pi}_i^a(\mathbf{x})\,|\,\rangle = -\mathrm{i}\,\frac{\delta}{\delta A_i^a(\mathbf{x})}\,\psi[A]. \tag{I.60b}$$

In the following, for the sake of simplicity the hat denoting the operator will be dropped, as long as no misunderstanding can arise. The Hamilton operator Eq. (I.57) in the Schrödinger picture described in Eqs. (I.60) leads to the functional Schrödinger equation

$$\int \mathrm{d}^d x \left[ -\frac{1}{2}\, \frac{\delta}{\delta A_i^a(\mathbf{x})}\, \frac{\delta}{\delta A_i^a(\mathbf{x})} + \frac{1}{4}\, F_{ij}^a(\mathbf{x})\, F_{ij}^a(\mathbf{x}) \right] \psi[A] = E\,\psi[A]. \tag{I.61}$$

The Heisenberg picture of quantum mechanics is, in general, more elegant than the Schrödinger picture: for example, adding a constant term to the Hamiltonian does not influence the equations of motion in the Heisenberg picture, while it appears as a phase factor in the Schrödinger formulation. It may then look as if the Schrödinger formulation had unnecessary complications; however, the equations of motion are easier to solve in the Schrödinger than in the Heisenberg picture.

### I.3.2 Gauss's law and gauge invariance

As already pointed out, the choice of the temporal gauge makes Gauss's law disappear from the set of the equations of motion. The theory based only on the Schrödinger equation with the Hamilton operator Eq. (I.57) is in some sense larger than it would be expected. To have a consistent classical limit, however, Gauss's law shall be imposed as a constraint: this choice will turn out to be motivated by considerations on gauge invariance.

Consider the operator which, in the quantized theory, expresses Gauss's law

$$\mathcal{G}^a(\mathbf{x}) = \hat{D}_i^{ab}(\mathbf{x})\,\Pi_i^b(\mathbf{x}) + g\rho_{\mathrm{ext}}^a(\mathbf{x}). \tag{I.62}$$

A straightforward calculation shows, with Eq. (I.56), that these operators satisfy the same commutation algebra as the underlying group generators

$$\big[\mathcal{G}^a(\mathbf{x}), \mathcal{G}^b(\mathbf{y})\big] = \mathrm{i}\, f^{abc}\, \mathcal{G}^c(\mathbf{x})\, \delta(\mathbf{x}-\mathbf{y}). \tag{I.63}$$

Moreover, the Gauss operators Eq. (I.62) commute with the Hamiltonian Eq. (I.57),

$$\big[H, \mathcal{G}^a(\mathbf{x})\big] = 0. \tag{I.64}$$

This implies that both operators can be simultaneously diagonalized, and if a physical state fulfills Gauss's law at one time, this will not change in the time evolution generated by the Hamiltonian Eq. (I.57). Gauss's law is then imposed on physical states by requiring

$$\mathcal{G}^a(\mathbf{x})\,|\text{phys}\rangle = 0, \tag{I.65}$$

i.e. physical states are annihilated by the Gauss operators.[5]

This request is closely connected to gauge invariance. As pointed out after Eq. (I.26), the pure gauge theory with external sources breaks gauge invariance, and one should either consider the whole theory, comprising dynamical fermions, or confine oneself to the pure gauge sector without external charges. To keep the following observations simple, only the case $\rho^a_{\text{ext}} = 0$ will be treated in the present section.

In the temporal gauge, there is still the freedom to perform space-dependent gauge transformations of the form

$$A_i(\mathbf{x}) \to \frac{\mathrm{i}}{g}\, U(\mathbf{x})\big(D_i U^\dagger(\mathbf{x})\big), \qquad U(\mathbf{x}) = \exp\big[\mathrm{i}t^a\varphi^a(\mathbf{x})\big], \tag{I.66}$$

which in infinitesimal form reads

$$A^a_i(\mathbf{x}) \to A^a_i(\mathbf{x}) + \frac{1}{g}\,\hat{D}^{ab}_i\varphi^b(\mathbf{x}) + \cdots \tag{I.67}$$

A unitary operator $\mathcal{U}$ which implements this transformation must satisfy

$$\mathcal{U}\,A_i(\mathbf{x})\,\mathcal{U}^\dagger \stackrel{!}{=} \frac{\mathrm{i}}{g}\,U(\mathbf{x})\big(D_i U(\mathbf{x})^\dagger\big). \tag{I.68}$$

Expanding $\mathcal{U}$ around the identity operator and comparing Eq. (I.68) with the infinitesimal transformation given in Eq. (I.67), it is simple to see that

$$\mathcal{U}_0 = \exp\left\{\frac{\mathrm{i}}{g}\int \mathrm{d}^d x\,\big[\hat{D}^{ab}_i\varphi^b(\mathbf{x})\big]\,\Pi^a_i(\mathbf{x})\right\} \tag{I.69}$$

is a possible realization. The meaning of the subscript '0' attached to the operator will be explained in the next section. Integrating by parts, and further assuming that the arising surface term can be discarded, Eq. (I.69) can be rewritten as

$$\mathcal{U}_0 = \exp\left\{-\frac{\mathrm{i}}{g}\int \mathrm{d}^d x\,\varphi^a(\mathbf{x})\big[\hat{D}^{ab}_i\,\Pi^b_i(\mathbf{x})\big]\right\} = \exp\left\{-\frac{\mathrm{i}}{g}\int \mathrm{d}^d x\,\varphi^a(\mathbf{x})\,\mathcal{G}^a_0(\mathbf{x})\right\}, \tag{I.70}$$

where $\mathcal{G}^a_0(\mathbf{x})$ is the Gauss operator as given in Eq. (I.62) with the external charges set to zero. The request Eq. (I.65) that the physical states are annihilated by the Gauss operators is equivalent, by Eq. (I.70), to demanding

$$\mathcal{U}_0\,|\text{phys}\rangle = |\text{phys}\rangle, \tag{I.71}$$

[5] Since the different colour components do not commute with each other, see Eq. (I.63), this is the only possible eigenvalue equation.

i.e. that the physical states are gauge invariant.

There are, of course, other ways to reach the same conclusion. Starting from the invariance of the Hamiltonian Eq. (I.57) under time-independent gauge transformations Eq. (I.66), the correlated Noether current can be derived: the operator $\mathcal{G}_0^a$ is the corresponding charge. Another possible viewpoint starts from a Taylor expansion of the state functional

$$\begin{aligned}\psi[A^U] &= \psi[A + \tfrac{1}{g}\hat{D}\varphi + \cdots] = \psi[A] + \int \mathrm{d}^d x\, \frac{1}{g}\, \hat{D}_i^{ab}\varphi^b(\mathbf{x})\, \frac{\delta\psi[A]}{\delta A_i^a(\mathbf{x})} + \cdots \\ &= \psi[A] - \frac{\mathrm{i}}{g}\int \mathrm{d}^d x\, \varphi^a(\mathbf{x})\, \hat{D}_i^{ab}\, \Pi_i^a(\mathbf{x})\, \psi[A] + \cdots \\ &= \psi[A] - \frac{\mathrm{i}}{g}\int \mathrm{d}^d x\, \varphi^a(\mathbf{x})\, \mathcal{G}_0^a(\mathbf{x})\, \psi[A] + \cdots \end{aligned} \tag{I.72}$$

(again under the assumption that the surface term arising in the integration by parts can be discarded), so that demanding gauge invariance of the state functional leads to the Gauss law constraint Eq. (I.65).

### I.3.3 Topological structure of the QCD vacuum

To keep the discussion simple, in this section only the physical case $d = 3$ and the $SU(2)$ group will be considered.

In the previous section, the issue of residual gauge invariance in the canonically quantized theory has been discussed. No assumption about the form of the gauge fields or of the transformation functions has been made. Here, all physically relevant field configurations are presumed to fall off faster than $1/|\mathbf{x}|$ at spatial infinity

$$\mathbf{A}(|\mathbf{x}| \to \infty) \sim \frac{1}{|\mathbf{x}|^{1+\varepsilon}}\,, \qquad \varepsilon > 0, \tag{I.73}$$

which corresponds to the restriction to fields with a finite classical action. Furthermore, allowed gauge transformations are supposed to approach a unique value at spatial infinity,

$$U(|\mathbf{x}| \to \infty) \to U_\infty\,. \tag{I.74}$$

There are merely plausibility reasons (but no proofs) for this choice [14]; for example, the total colour charge would be ill defined for more general, direction-dependent gauge transformations.

The boundary condition Eq. (I.74) implies a compactification of the three-dimensional configuration space $\mathbb{R}^3$ into the 3-sphere $S^3$. The gauge transformations $U$ which satisfy these conditions fall into different homotopy classes, which can be labelled by an integer $n[U]$, the *winding number*

$$n[U] = \frac{\mathrm{i}}{24\pi^2}\int \mathrm{d}^3 x\, \varepsilon_{ijk}\ \mathrm{tr}\big[U(\partial_i U^\dagger)U(\partial_j U^\dagger)U(\partial_k U^\dagger)\big]. \tag{I.75}$$

Field configurations with different winding numbers are not homotopically equivalent, i.e. they cannot be deformed continuously into each other without violating the boundary condition Eq. (I.74).[6] From this, another argument supporting the assumption Eq. (I.74)

[6] Two gauge transformations $U_1(\mathbf{x})$ and $U_2(\mathbf{x})$ are homotopically equivalent if there exists a continuous mapping $U(\mathbf{x}, a)$ such that $U(\mathbf{x}, 0) = U_1(\mathbf{x})$ and $U(\mathbf{x}, 1) = U_2(\mathbf{x})$, and such that the boundary conditions are satisfied for every $a$.

about the possible transformations follows: without the boundary condition Eq. (I.74), the integral Eq. (I.75) may diverge. A representative for the $SU(2)$ transformations with unity winding number is

$$U_{n=1}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\pi f(|\mathbf{x}|)\boldsymbol{\sigma}\cdot\hat{\mathbf{x}}}, \quad \text{with} \quad f(0)=0, \quad f(\infty)=1, \tag{I.76}$$

($\boldsymbol{\sigma}$ stands for the Pauli matrices) while a representative of the $N$-th class is

$$U_{n=N}(\mathbf{x}) = \big[U_{n=1}(\mathbf{x})\big]^N. \tag{I.77}$$

Gauge transformations which can be homotopically deformed to unity belong to the homotopy class with vanishing winding number, and are referred to as 'small' gauge transformations. These are the only transformations which can be obtained by iterating an infinitesimal one, and the generator of these transformations is the Gauss operator Eq. (I.62), as seen in the previous section. This explains the subscript '0' which has been attached to the operator $\mathcal{U}_0$ [Eq. (I.69)].

Nevertheless, the Hamilton operator Eq. (I.57) is invariant under *all* gauge transformations, small or large, and physical states can change by a phase factor at most. For transformations with unity winding number, a state functional will then become

$$\mathcal{U}\psi[\mathbf{A}] = \mathrm{e}^{-\mathrm{i}\theta}\,\psi[\mathbf{A}]\,, \qquad \text{for } n[U]=1, \tag{I.78}$$

where $\theta$ is an undetermined parameter. Furthermore, from Eqs. (I.76) and (I.77) it can be inferred that the correct generalization of Eq. (I.71) reads

$$\mathcal{U}\psi[\mathbf{A}] = \mathrm{e}^{-\mathrm{i}\theta n[U]}\,\psi[\mathbf{A}]. \tag{I.79}$$

It can be shown that there is no way to avoid the introduction of this new parameter in the theory. Also assuming that physical states are indeed invariant under all gauge transformations, a $\theta$ dependence can follow from the ambiguity of the Lagrangian, to which a term of the form

$$\theta\,\frac{g^2}{64\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,F^a_{\mu\nu}\,F^a_{\rho\sigma} = -\theta\,\frac{g^2}{16\pi^2}\,\varepsilon_{ijk}\,E^a_i\,F^a_{jk} = -\theta\,\frac{g^2}{8\pi^2}\,E^a_i\,B^a_i \tag{I.80}$$

can be added (the numerical factor is a matter of convenience). This term can be written as a total divergence

$$\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}\,F^a_{\rho\sigma} = 4\,\partial_\mu K^\mu, \tag{I.81}$$

where the *topological current*

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}\left(A^a_\nu\partial_\rho A^a_\sigma - \tfrac{g}{3}\,f^{abc}A^a_\nu A^b_\rho A^c_\sigma\right), \tag{I.82}$$

has been introduced. Thus, the classical Hamilton function Eq. (I.38) retains its form; what does change is the relation Eq. (I.37b) between the canonical momentum and the electric field, which turns into

$$\Pi^a_i = -E^a_i + \theta\,\frac{g^2}{16\pi^2}\,\varepsilon_{ijk}F^a_{jk} = -E^a_i + \theta\,\frac{g^2}{8\pi^2}\,B^a_i, \tag{I.83}$$

and the functional Schrödinger equation (I.61) becomes

$$\int \mathrm{d}^3x\left[\frac{1}{2}\left(\frac{\delta}{\mathrm{i}\delta A^a_i(\mathbf{x})} - \theta\,\frac{g^2}{16\pi^2}\,B^a_i(\mathbf{x})\right)^2 + \frac{1}{2}\,B^a_i(\mathbf{x})\,B^a_i(\mathbf{x})\right]\Psi[\mathbf{A}] = E\,\Psi[\mathbf{A}], \tag{I.84}$$

where the states $\Psi[\mathbf{A}]$ are gauge invariant under all gauge transformations. Writing the gauge invariant functional $\Psi[\mathbf{A}]$ as

$$\Psi[\mathbf{A}] = \mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}]}\,\psi[\mathbf{A}] \tag{I.85}$$

with some functional $W[\mathbf{A}]$ satisfying

$$\frac{\delta W[\mathbf{A}]}{\delta A_i^a(\mathbf{x})} = \frac{g^2}{8\pi^2}\,B_i^a(\mathbf{x}), \tag{I.86}$$

the additional $\theta$-dependent terms can be eliminated and obtains again Eq. (I.61). Equation (I.86) is easily integrated, and the result

$$W[\mathbf{A}] = \frac{g^2}{16\pi^2}\int \mathrm{d}^3x\,\varepsilon_{ijk}\big[A_i^a\partial_j A_k^a + \tfrac{g}{3}\,f^{abc}A_i^aA_j^bA_k^c\big] = \frac{g^2}{16\pi^2}\int \mathrm{d}^3x\,K^0(\mathbf{x}) \tag{I.87}$$

is called the *Chern–Simons action*. Under a gauge transformation fulfilling the boundary condition Eq. (I.74), the Chern–Simons action behaves like

$$W[\mathbf{A}^U] = W[\mathbf{A}] + n[U], \tag{I.88}$$

which confirms the transformation property Eq. (I.79) of the functional $\psi[\mathbf{A}]$ in Eq. (I.85). Furthermore, acting with the operator $\mathcal{U}_0$ as given in Eq. (I.69) [or Eq. (I.70)] on the Chern–Simons action Eq. (I.87), gives

$$\mathcal{U}_0 W[\mathbf{A}] = W[\mathbf{A}], \tag{I.89}$$

which shows again how the explicit realization Eq. (I.69) [(I.70)] can describe only small transformations.

No a-priori computation for the vacuum angle $\theta$ is available; physical effects involve a non-vanishing neutron electric dipole moment [15], and the measurements indicate that $\theta$ should be smaller than $10^{-10}$. It is also worth mentioning that there is no unanimous agreement about the fact that the parameter $\theta$ appearing in the transformation property Eq. (I.79) is the same $\theta$ occurring in the additional term in the Lagrangian, Eq. (I.80). Giving up the assumptions Eqs. (I.73) and (I.74), the first one can be proven to be indeed zero, while nothing can be said about the second one.

However, the fact that both parameters lead to the same Hamiltonian, and consequently to the same physical effects, seems to legitimate the use of Occam's razor: since the same physics is obtained in both cases, no matter how this arbitrary parameter is introduced, it seems appropriate to speak about only one $\theta$ angle.

Physical effects of the $\theta$-vacuum are dealt with in Article 5.

## I.4 The Yang–Mills Hamiltonian in Coulomb gauge

### I.4.1 Gauge fixing and the Gribov problem

As previously discussed, the Weyl gauge does not fix the fields completely, since there exists always a freedom to perform time-independent gauge transformations $U(\mathbf{x})$. Moreover, as already noticed in the previous section, if the choice of the temporal gauge gets rid of the troubles related to the vanishing momentum conjugated to $A_0$, it makes nevertheless

Gauss's law disappear from the set of equations of motion. As a consequence, Gauss's law has to be imposed as an external constraint on the physical states, with the meaning that physical states must be gauge invariant, at least under small gauge transformations.

Working with gauge invariant states can be awkward, mostly when using a variational approach, which is one of the subjects of this work. It is then convenient to fix the gauge by explicitly resolving Gauss's law, and the Coulomb gauge $\partial_i A_i^a = 0$ is well suited to this purpose. The Coulomb gauge is of course not Lorentz invariant, but explicit invariance has been lost by using the Hamiltonian formulation of the theory and by choosing the temporal gauge anyway.

Physical quantities are invariant under a gauge transformation of the fields, i.e. all fields lying on the *gauge orbit*

$$\mathscr{O}_A = \left\{ \mathbf{A}^U : \mathbf{A}^U = U\mathbf{A}U^\dagger + \frac{\mathrm{i}}{g} U \boldsymbol{\nabla} U^\dagger \right\} \tag{I.90}$$

are, in fact, equivalent. A gauge fixing is (or rather should be) a choice which completely defines the field. A standard method to implement a gauge is the procedure developed by Faddeev and Popov [16] to handle path integrals in gauge theories. A particular gauge can be enforced in the scalar product of the Hilbert space by inserting the functional identity

$$1 = \int \mathcal{D}U \; \delta\big(\chi[A^U]\big) \; \mathrm{Det}\left( \frac{\delta \chi[A^U]}{\delta \varphi} \right), \tag{I.91}$$

where $\chi[A]$ is the gauge condition and the integral is performed over the gauge group. Equation (I.91) can be interpreted as the functional generalization of

$$1 = \int \mathrm{d}x \, |f'(x)| \; \delta\big(f(x)\big). \tag{I.92}$$

The operator in the functional determinant of Eq. (I.91) is for the Coulomb gauge ($\chi[A] = \partial_i A_i^a$)

$$\frac{\delta}{\delta \varphi^b(\mathbf{x})} \, \partial_i \big(A_i^U(\mathbf{y})\big)^a = \frac{1}{g} \, \partial_i \hat{D}_i^{ab}(\mathbf{x}) \, \delta(\mathbf{x} - \mathbf{y}), \tag{I.93}$$

and Eq. (I.91) can be rewritten as

$$1 = \int \mathcal{D}U \; \delta\big(\partial_i A_i^U\big) \; \mathcal{J}_{A^U} \, , \qquad \mathcal{J}_{A^U} \equiv \mathrm{Det}\big(-\partial_i \hat{D}_i^{ab}[A^U]\big), \tag{I.94}$$

where we introduced the *Faddeev–Popov determinant* $\mathcal{J}_A$.

Before the implementation of the gauge is carried out, the issues related to the Gribov ambiguity for non-abelian gauge fields should be briefly reviewed. Equation (I.92) is known to be valid only if the function $f(x)$ has a single root. Correspondingly, for the functional identity Eq. (I.94) to be correct, the transversality condition $\partial_i A_i^a = 0$ should have a unique solution. As seen in Sec. I.2.2, this can be guaranteed in the abelian theory by imposing suitable boundary conditions. However, as pointed out by Gribov [17], this is not the case for non-abelian gauge theories: there exists always so-called *gauge copies*, i.e. transverse field configurations which are connected by a gauge transformation. Even the choice of boundary conditions does not eliminate the gauge copies.

In order to pick out only one representative from each gauge orbit, the functional integration should be restricted to the set of gauge inequivalent configurations. A possible

way to choose this set for the Coulomb gauge is to consider the following function on the gauge orbit[7]

$$F_A[U] = \int \mathrm{d}^d x \, (A^U(\mathbf{x}))^a_i \, (A^U(\mathbf{x}))^a_i \,, \tag{I.95}$$

and expand it around the identity transformation [see Eq. (I.67)]

$$\begin{aligned} F_A[U] = \int \mathrm{d}^d x \, A^a_i(\mathbf{x}) \, A^a_i(\mathbf{x}) - \frac{2}{g} \int \mathrm{d}^d x \, \varphi^a(\mathbf{x}) \, \partial_i A^a_i(\mathbf{x}) + \\ + \frac{1}{g^2} \int \mathrm{d}^d x \, \varphi^a(\mathbf{x}) \big( -\partial_i \hat{D}^{ab}_i \varphi^b(\mathbf{x}) \big) + \mathcal{O}(\varphi^3). \end{aligned} \tag{I.96}$$

The minima of this function are transverse field configurations $\partial_i A^a_i = 0$ with a positive definite Faddeev–Popov operator, $-\partial_i \hat{D}_i \geq 0$. The set of all local minima

$$\Omega := \big\{ A : \partial_i A_i = 0 \,, \; -\partial_i \hat{D}_i[A] \geq 0 \big\} \tag{I.97}$$

defines the *first Gribov region*. This is bounded in every direction and convex, and the Faddeev–Popov operator acquires the eigenvalue zero on its boundary $\partial\Omega$. This implies that the Faddeev–Popov determinant $\mathcal{J}_A$ vanishes on $\partial\Omega$.

Unfortunately, the Gribov region $\Omega$ is still not free of gauge copies [19]. The configuration space should be restricted to the set of the *absolute* minima

$$\Lambda := \big\{ A : F_A[\mathbb{1}] \leq F_A[U] \quad \forall U \big\} \tag{I.98}$$

which is called *fundamental modular region* (FMR) $\Lambda$. It can be shown that the FMR is a proper subset of the first Gribov region, $\Lambda \subset \Omega$, that it has a non-trivial topological structure, that both $\Lambda$ and $\Omega$ are bounded in every direction and convex, and that they have some points in common. The FMR is then almost free of Gribov copies, since some absolute minima can be degenerate.

Unfortunately, no feasible way to introduce the explicit restriction to the FMR is available yet in the continuum.[8] It has been argued [20] that the functional integration over these two regions should yield the same result. In the following, this conjecture will be relied on, and the connected subtleties will be ignored. The functional identity Eq. (I.94), when restricted to the Gribov region, still retains its validity, and will be used in the next section to fix the Coulomb gauge.

### I.4.2 Implementation of the Coulomb gauge

As already pointed out, the physical problem consists in solving the functional Schrödinger equation (I.61) supplemented by the constraint of Gauss's law Eq. (I.65). There are various methods for the implementation of the Coulomb gauge: among others, constrained quantization by means of Dirac's brackets (see e.g. Ref. [21] and references therein), and an explicit elimination of the redundant degrees of freedom [22]. Here, the Coulomb gauge will be implemented in the scalar product by means of the Faddeev–Popov method as described in the previous section. This derivation is much more simple to understand, at least ignoring all kind of mathematical troubles one runs into.

[7] This presentation does not follow Gribov's original line of thought, but it has been put forward here for pedagogical reasons; an introductory discussion can be found in Ref. [18].

[8] With the exception of the theory in $1+1$ dimensions.

It should be stressed again that the quantization has been already performed in the temporal gauge, and merely a way to implement an additional constraint on the physical states will be discussed.

In the Coulomb gauge the dynamical variables are the transverse field configurations $A^{\perp}$. The commutation relations (I.56) become in the transverse subspace

$$\left[A_i^{\perp a}(\mathbf{x}), \Pi_j^{\perp b}(\mathbf{y})\right] = \mathrm{i}\,\delta^{ab}\, t_{ij}(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}), \tag{I.99}$$

where $t_{ij}$ is the transverse projector

$$t_{ij}(\mathbf{x}) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \left(\delta_{ij} - \frac{p_i\, p_j}{\mathbf{p}^2}\right). \tag{I.100}$$

In order to proceed, Gauss's law (with external charges reintroduced)

$$\hat{D}_i^{ab}\, \Pi_i^b\, \psi[A] = -g\rho_{\mathrm{ext}}^a\, \psi[A], \tag{I.101}$$

shall be explicitly resolved. To this purpose, the momentum operator is splitted in a longitudinal and transverse part $\Pi_i^a = \Pi_i^{\perp a} + \partial_i \xi^a$, and with the transversality condition $\partial_i \Pi_i^{\perp} = 0$ Eq. (I.101) becomes

$$-g\rho_{\mathrm{ext}}^a\, \psi[A] = \big(\hat{D}_i^{ab}\, \Pi_i^{\perp b} + \hat{D}_i^{ab}\, \partial_i \xi^b\big)\psi[A] = \big(g\hat{A}_i^{ab}\, \Pi_i^{\perp b} + \hat{D}_i^{ab}\, \partial_i \xi^b\big)\psi[A], \tag{I.102}$$

which can be solved for the longitudinal component of the momentum operator as

$$-\hat{D}_i^{ab}\, \partial_i\, \xi^b\, \psi[A] = g\rho^a\, \psi[A]. \tag{I.103}$$

In Eq. (I.103) the total colour charge density

$$\rho^a(\mathbf{x}) = \rho_{\mathrm{ext}}^a(\mathbf{x}) + \hat{A}_i^{\perp ab}(\mathbf{x})\, \Pi_i^{\perp b}(\mathbf{x}) \tag{I.104}$$

has been defined. Notice that only the transverse part of the gauge field enters the total charge density. Introducing now the Green function $G_A$ of the Faddeev–Popov operator

$$\big(-\delta^{ab}\partial^2 - \hat{A}_i^{ab}(\mathbf{x})\partial_i\big)\, G_A^{bc}(\mathbf{x},\mathbf{y}) = \delta^{ac}\,\delta(\mathbf{x}-\mathbf{y}), \tag{I.105}$$

the action of the momentum operator on the state functional can finally be written as

$$\Pi_i^a(\mathbf{x})\,\psi[A] = \big(\Pi_i^{\perp a}(\mathbf{x}) + \partial_i \xi^a(\mathbf{x})\big)\psi[A], \qquad \xi^a(\mathbf{x}) = g\int \mathrm{d}^d y\, G_A^{ab}(\mathbf{x},\mathbf{y})\, \rho^b(\mathbf{y}). \tag{I.106}$$

The eigenstates of the Hamiltonian Eq. (I.57), i.e. the solutions of the functional Schrödinger equation (I.61), span a Hilbert space $\mathscr{H}$ which is larger that the physical one. The scalar product in $\mathscr{H}$ is defined by

$$\langle\phi|\, O[A,\Pi]\,|\psi\rangle = \int \mathcal{D}A\, \phi^*[A]\, O[A, -\mathrm{i}\tfrac{\delta}{\delta A}]\, \psi[A]\,. \tag{I.107}$$

When restricted to physical states, the integration in Eq. (I.107) is ill defined, since the integral extends over all gauge equivalent configurations. This simply means that physical

states are not normalizable in $\mathscr{H}$. In order to restrict Eq. (I.107) to the physical states, the following expression should be considered

$$\langle\phi|\, O[A,\Pi]\,|\psi\rangle = \left[\int \mathcal{D}U\right]^{-1} \int \mathcal{D}A\; \phi^*[A]\; O[A, -\mathrm{i}\tfrac{\delta}{\delta A}]\; \psi[A]\,, \tag{I.108}$$

where the normalization constant is the integral over the gauge group (see e.g. Ref. [23] for a discussion). Equation (I.108) is a $\infty/\infty$ form, and has to be defined by means of the Faddeev–Popov method. In the case of the Hamilton operator, it is convenient to first perform an integration by parts in the electric term before inserting the identity Eq. (I.94) into Eq. (I.108). One can then write ($\mathcal{N} = (\int \mathcal{D}U)^{-1}$)

$$\begin{aligned}
&\langle\phi|\, H\,|\psi\rangle = \\
&= \mathcal{N} \int \mathcal{D}A\; \phi^*[A] \big\{ \tfrac{1}{2}\,\Pi^2 + \tfrac{1}{4}(F_{ij})^2 \big\} \psi[A] \\
&= \mathcal{N} \int \mathcal{D}A\; \big\{ \tfrac{1}{2}(\Pi\phi[A])^*(\Pi\psi[A]) + \tfrac{1}{4}(F_{ij})^2 \phi^*[A]\psi[A] \big\} \\
&= \mathcal{N} \int \mathcal{D}A \int \mathcal{D}U\; \delta\big(\partial_i A_i^U\big)\; \mathcal{J}_{A^U}\; \big\{ \tfrac{1}{2}(\Pi\phi[A])^*(\Pi\psi[A]) + \tfrac{1}{4}(F_{ij})^2 \phi^*[A]\psi[A] \big\} \\
&= \mathcal{N} \int \mathcal{D}A \int \mathcal{D}U\; \delta\big(\partial_i A_i^U\big)\; \mathcal{J}_{A^U}\; \big\{ \tfrac{1}{2}(\Pi^U\phi[A^U])^*(\Pi^U\psi[A^U]) + \tfrac{1}{4}(F_{ij}^U)^2 \phi^*[A^U]\psi[A^U] \big\}.
\end{aligned} \tag{I.109}$$

In the last passage, the gauge invariance of the states and the Hamilton operator in temporal gauge have been used. Assuming the invariance of the functional measure, $\mathcal{D}A^U = \mathcal{D}A$,[9] the integration variable can be shifted, $A^U \to A$, and the integral over the gauge group $\int \mathcal{D}U$ can be factorized out of the integral and canceled with the normalization. This results in

$$\langle\phi|\, H\,|\psi\rangle = \int \mathcal{D}A\; \delta\big(\partial_i A_i\big)\; \mathcal{J}_A\; \big\{ \tfrac{1}{2}(\Pi\phi[A])^*(\Pi\psi[A]) + \tfrac{1}{4}(F_{ij})^2 \phi^*[A]\psi[A] \big\}. \tag{I.110}$$

The reason for the integration by parts in the electric term of the Hamiltonian is that Gauss's law tells how to express the functional derivative of state functionals in terms of transverse quantities only, see Eq. (I.106). The first term in Eq. (I.110) can thus be written as

$$\begin{aligned}
&\int \mathcal{D}A\; \delta\big(\partial_i A_i\big)\; \mathcal{J}_A \int \mathrm{d}^d x\; (\Pi_i^a(\mathbf{x})\phi[A])^*(\Pi_i^a(\mathbf{x})\psi[A]) = \\
&\quad = \int \mathcal{D}A \int \mathrm{d}^d x\; \delta(\partial_i A_i)\; \mathcal{J}_A \\
&\qquad\qquad \times \Big[ \big(\Pi_i^{\perp a}(\mathbf{x}) + \partial_i \xi^a(\mathbf{x})\big)\phi[A] \Big]^* \Big[ \big(\Pi_i^{\perp a}(\mathbf{x}) + \partial_i \xi^a(\mathbf{x})\big)\psi[A] \Big] = \\
&\quad = \int \mathcal{D}A \int \mathrm{d}^d x\; \delta(\partial_i A_i)\; \mathcal{J}_A \\
&\qquad\qquad \times \Big[ \big(\Pi_i^{\perp a}(\mathbf{x})\;\phi[A]\big)^* \big(\Pi_i^{\perp a}(\mathbf{x})\;\psi[A]\big) + \big(\partial_i \xi^a(\mathbf{x})\phi[A]\big)^* \big(\partial_i \xi^a(\mathbf{x})\psi[A]\big) \Big]\,.
\end{aligned} \tag{I.111}$$

[9] This kind of manipulations is somewhat formal in the continuum theory, but can be well defined in the lattice formulation.

In the last line it has been used the fact that the mixed terms (longitudinal times transverse components of the momentum operator) vanish, as it is easily seen performing an integration by parts, e.g.

$$\int \mathrm{d}^d x \left(\Pi_i^{\perp a}(\mathbf{x})\,\phi^*[A]\right)\left(\partial_i \xi^a(\mathbf{x})\,\psi[A]\right) = -\int \mathrm{d}^d x \left(\underbrace{\partial_i\,\Pi_i^{\perp a}(\mathbf{x})}_{=0}\,\phi^*[A]\right)\left(\xi^a(\mathbf{x})\,\psi[A]\right). \quad \text{(I.112)}$$

By using $\Pi^{\perp *} = -\Pi^{\perp}$, performing a functional integration by parts, and under the assumption that the surface term vanishes,[10] the first term in the brackets in Eq. (I.111) can be written as

$$\begin{aligned}
\int \mathcal{D}A \int \mathrm{d}^d x\, \delta(\partial_i A_i)\, \mathcal{J}_A \left(\Pi_i^{\perp a}(\mathbf{x})\,\phi[A]\right)^* \left(\Pi_i^{\perp a}(\mathbf{x})\,\psi[A]\right) = \\
= \int \mathcal{D}A^{\perp} \int \mathrm{d}^d x\, \mathcal{J}_{A^{\perp}} \left(\Pi_i^{\perp a}(\mathbf{x})\,\phi[A^{\perp}]\right)^* \left(\Pi_i^{\perp a}(\mathbf{x})\,\psi[A^{\perp}]\right) = \\
= \int \mathcal{D}A^{\perp} \int \mathrm{d}^d x\, \mathcal{J}_{A^{\perp}} \left(-\Pi_i^{\perp a}(\mathbf{x})\,\phi^*[A^{\perp}]\right) \left(\Pi_i^{\perp a}(\mathbf{x})\,\psi[A^{\perp}]\right) = \\
= \int \mathcal{D}A^{\perp} \int \mathrm{d}^d x\, \phi^*[A^{\perp}]\, \Pi_i^{\perp a}(\mathbf{x}) \mathcal{J}_{A^{\perp}}\, \Pi_i^{\perp a}(\mathbf{x})\,\psi[A^{\perp}]. \qquad \text{(I.113)}
\end{aligned}$$

In order to obtain matrix elements in the physical space in the form

$$\langle \phi |\, O[A,\Pi]\,| \psi \rangle = \int \mathcal{D}A^{\perp}\, \mathcal{J}_{A^{\perp}}\, \phi^*[A^{\perp}]\, O[A^{\perp}, \Pi^{\perp}]\, \psi[A^{\perp}]\,, \qquad \text{(I.114)}$$

the operator coming from the transverse part of the momentum $\Pi$ is identified as

$$H_k = \frac{1}{2} \int \mathrm{d}^d x\, \mathcal{J}_{A^{\perp}}^{-1}\, \Pi_i^{\perp a}(\mathbf{x})\, \mathcal{J}_{A^{\perp}}\, \Pi_i^{\perp a}(\mathbf{x})\,. \qquad \text{(I.115)}$$

In analogy to classical mechanics, Eq. (I.115) is often referred to as 'kinetic' operator.

The second term in Eq. (I.111) is slightly more complicated: inserting the result Eq. (I.106) for the longitudinal component of the momentum operator yields

$$\begin{aligned}
\int \mathcal{D}A \int \mathrm{d}^d x\, \delta(\partial_i A_i)\, \mathcal{J}_A \left(\partial_i \xi^a(\mathbf{x}) \phi[A]\right)^* \left(\partial_i \xi^a(\mathbf{x}) \psi[A]\right) = \\
= \int \mathcal{D}A \int \mathrm{d}^d x\, \delta(\partial_i A_i)\, \mathcal{J}_A \left(\xi^a(\mathbf{x}) \phi[A]\right)^* \left((-\partial^2) \xi^a(\mathbf{x}) \psi[A]\right) = \\
= g^2 \int \mathcal{D}A^{\perp} \int \mathrm{d}^d x\; \mathrm{d}^d y\; \mathrm{d}^d z\, \mathcal{J}_{A^{\perp}}\, G^{ab}_{A^{\perp}}(\mathbf{x},\mathbf{y}) \left(\rho^{b^*}(\mathbf{y})\,\phi^*[A^{\perp}]\right) \\
\times (-\partial_x^2)\, G^{ac}_{A^{\perp}}(\mathbf{x},\mathbf{z})\, \rho^c(\mathbf{z})\, \psi[A^{\perp}]. \qquad \text{(I.116)}
\end{aligned}$$

The external colour charge density is real, while complex conjugation of the gauge field charge density leads to a sign change,

$$(\rho^b)^* = \rho^b_{\mathrm{ext}} - \hat{A}_i^{\perp bd}\, \Pi_i^{\perp d}. \qquad \text{(I.117)}$$

[10] This is the case when the integration is restricted to the first Gribov region, on whose border the Faddeev–Popov determinant $\mathcal{J}_{A^{\perp}}$ vanishes.

Performing, as in Eq. (I.113), a functional integration by parts yields a further minus sign, and since the transverse momentum operator commutes with the field in the adjoint representation (due to the presence of the totally antisymmetric structure constants), Eq. (I.116) can be rewritten as

$$\begin{aligned}&\int \mathcal{D}A \int \mathrm{d}^d x\, \delta(\partial_i A_i)\, \mathcal{J}_A \left(\partial_i \xi^a(\mathbf{x}) \phi[A]\right)^* \left(\partial_i \xi^a(\mathbf{x}) \psi[A]\right) = \\ &= g^2 \int \mathcal{D}A^\perp \int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathrm{d}^d z\, \phi^*[A^\perp]\, \rho^b(\mathbf{y})\, \mathcal{J}_{A^\perp} G^{ab}_{A^\perp}(\mathbf{x},\mathbf{y}) (-\partial_x^2) G^{ac}_{A^\perp}(\mathbf{x},\mathbf{z})\, \rho^c(\mathbf{z})\, \psi[A^\perp].\end{aligned} \tag{I.118}$$

Introducing now the Coulomb interaction kernel

$$F^{ab}_{A^\perp}(\mathbf{x},\mathbf{y}) := \int \mathrm{d}^d z\, G^{ca}_{A^\perp}(\mathbf{z},\mathbf{x}) \left(-\partial_z^2\right) G^{cb}_{A^\perp}(\mathbf{z},\mathbf{y}), \tag{I.119}$$

the contribution of the longitudinal momentum to the Hamilton operator can be written as

$$H_{\mathrm{C}} = \frac{g^2}{2} \int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathcal{J}^{-1}_{A^\perp}\, \rho^a(\mathbf{x})\, \mathcal{J}_{A^\perp}\, F^{ab}_{A^\perp}(\mathbf{x},\mathbf{y})\, \rho^b(\mathbf{y}), \tag{I.120}$$

which is called the 'Coulomb term', since it reduces to the usual, instantaneous Coulomb interaction between static charges in the abelian case.

Since the magnetic term [last term in Eq. (I.110)] does not contain derivative operators, the gauge fixing gives immediately

$$\begin{aligned}\int \mathcal{D}A\, \delta\left(\partial_i A_i\right)\, \mathcal{J}_A \int \mathrm{d}^d x\, F^a_{ij}(\mathbf{x})\, F^a_{ij}(\mathbf{x})\, \phi^*[A]\, \psi[A] = \\ = \int \mathcal{D}A^\perp\, \mathcal{J}_{A^\perp} \int \mathrm{d}^d x\, F^{\perp a}_{ij}(\mathbf{x})\, F^{\perp a}_{ij}(\mathbf{x})\, \phi^*[A^\perp]\, \psi[A^\perp],\end{aligned} \tag{I.121}$$

where the spatial part of the field strength tensor has to be evaluated with transverse fields only.

Collecting Eqs. (I.115), (I.120) and (I.121), and dropping the transversality symbol, the complete Hamiltonian in Coulomb gauge can be written as

$$\boxed{\begin{aligned}H_{\mathrm{YM}} = \frac{1}{2} \int \mathrm{d}^d x\, \mathcal{J}_A^{-1}\, \Pi^a_i(\mathbf{x})\, \mathcal{J}_A\, \Pi^a_i(\mathbf{x}) + \frac{1}{4} \int \mathrm{d}^d x\, F^a_{ij}(\mathbf{x})\, F^a_{ij}(\mathbf{x}) + \\ + \frac{g^2}{2} \int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathcal{J}_A^{-1}\, \rho^a(\mathbf{x})\, \mathcal{J}_A\, F^{ab}_A(\mathbf{x},\mathbf{y})\, \rho^b(\mathbf{y})\end{aligned}} \tag{I.122}$$

The abelian case, i.e. electrodynamics, can be treated in order to make the meaning of this expression more clear. The transition is made by setting the structure constants to zero, $f^{abc} = 0$, which also implies that the colour charge density of the gauge field [the second term in Eq. (I.104)] vanishes—the electromagnetic field carries no charge. The covariant derivative in the adjoint representation reduces to the normal one $\hat{D} \to \partial$, which also implies the Coulomb kernel to become

$$F^{ab}_A(\mathbf{x},\mathbf{y}) \to \left[(-\partial^2)^{-1}\, (-\partial^2)\, (-\partial^2)^{-1}\right]^{ab}_{\mathbf{x},\mathbf{y}} = \delta^{ab} \left[(-\partial^2)^{-1}\right]_{\mathbf{x},\mathbf{y}} \overset{d=3}{=} \frac{\delta^{ab}}{4\pi} \frac{1}{|\mathbf{x}-\mathbf{y}|}, \tag{I.123}$$

and the Faddeev–Popov determinant to reduce to the field-independent quantity

$$\mathcal{J}_A\big|_{f^{abc}=0} = \mathrm{Det}(-\partial^2). \tag{I.124}$$

The magnetic term also reduces to the usual squared curl of the vector potential; by substituting Eqs. (I.123) and (I.124) into Eq. (I.122), the Coulomb gauge Hamiltonian for the $U(1)$ gauge group becomes

$$H = \frac{1}{2}\int \mathrm{d}^3x \left[\mathbf{\Pi}(\mathbf{x})^2 + \big(\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x})\big)^2\right] + \frac{g^2}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\,\frac{\rho_{\mathrm{ext}}(\mathbf{x})\,\rho_{\mathrm{ext}}(\mathbf{y})}{4\,\pi\,|\mathbf{x}-\mathbf{y}|}\,, \tag{I.125}$$

which represents the total energy of an electromagnetic field with static external charges.

Since the Hamiltonian Eq. (I.122) does not change if the Faddeev–Popov determinant $\mathcal{J}_A$ [Eq. (I.94)] is multiplied by a constant, $\mathcal{J}_A$ can be rescaled

$$\mathcal{J}_A = \mathrm{Det}(-\hat{D}\partial) \longrightarrow \frac{\mathrm{Det}(-\hat{D}\partial)}{\mathrm{Det}(-\partial^2)} \tag{I.126}$$

so that $\mathcal{J}_A = 1$ for $g = 0$.

The $\theta$-dependent terms, discussed in Sec. I.3.3, have not been considered here. The implementation of the gauge in these terms is straightforward, and a short presentation is given in Article 5.

### I.4.3 Variational approach in the Schrödinger picture

The Coulomb gauge Hamilton operator Eq. (I.122) has been subject of intensive studies over the last few years [10–12]. In a variational approach, an ansatz depending on some parameters is chosen for the vacuum wave functional $\psi[A]$. These parameters are then determined through a variational principle by minimizing the energy functional,

$$E[\psi] = \int \mathcal{D}A\,\mathcal{J}_A\,\psi^*[A]\,H_{\mathrm{YM}}\big[A, -\mathrm{i}\tfrac{\delta}{\delta A}\big]\,\psi[A] \longrightarrow \min. \tag{I.127}$$

In particular, a Gaussian-type wave functional, depending on a variational kernel $\omega$, has been investigated [11,12]

$$\psi_\omega[A] = \mathcal{J}_A^{-1/2}\exp\left\{-\frac{1}{2}\int \frac{\mathrm{d}^dp}{(2\pi)^d}\,A_i^a(-\mathbf{p})\,\omega(\mathbf{p})\,A_i^a(\mathbf{p})\right\}. \tag{I.128}$$

Within this ansatz, the variational kernel $\omega(\mathbf{p})$ turns out to represent the inverse gluon propagator,

$$\Big\langle A_i^a(\mathbf{p})\,A_j^b(\mathbf{q})\Big\rangle = \int \mathcal{D}A\,\mathcal{J}_A\,|\psi_\omega[A]|^2\,A_i^a(\mathbf{p})\,A_j^b(\mathbf{q}) = \delta^{ab}\,\frac{t_{ij}(\mathbf{p})}{2\omega(\mathbf{p})}\,\delta(\mathbf{p}+\mathbf{q}). \tag{I.129}$$

The minimization of the energy density as given in Eq. (I.127) leads to a set of integral equations for the kernel $\omega(\mathbf{p})$ (the 'gap' equation), and for the ghost and the Coulomb form factors (discussed in Article 4). An infrared analysis of the equations [24] has shown the existence of two possible power law solutions for the gluon propagator, and both of them have been found numerically [11,12].

The numerical results obtained so far seem to grasp the essential features of the infrared physics. The obtained gluon energy (see Fig. 4.16, p. 109) approaches its perturbative form in the high-momentum regime, and diverges in the infrared. One of the two possible solutions [12] yields furthermore a linearly rising static quark potential, an infrared enhanced running coupling constant with no Landau pole [24], a topological susceptibility in accord with lattice data (see Article 5), a perimeter law for the 't Hooft loop [25], and, within an approximate Dyson equation, an area law for the spatial Wilson loop [26].

## I.5 Overview of the thesis

This thesis is presented in cumulative form. Rather than grouping the papers in publication order, it seemed more appropriate to arrange them in a logical pattern. The main consequence of this choice is that the oldest article (which presents an application of the general methods described in the previous ones) is the last in the sequence.

**Article 1** deals with an heat-kernel expansion of the Faddeev–Popov determinant. While in perturbative approaches it is common to introduce ghost fields and exploit BRST invariance to renormalize the theory, in the Hamiltonian approach the Faddeev–Popov determinant is treated non-perturbatively. The heat kernel allows one to identify the divergences, which can be removed by introducing appropriate counterterms.

**Article 2** contains a perturbative analysis of propagators in pure Yang–Mills theory. By means of standard Rayleigh–Schrödinger perturbation theory, the corrections to the vacuum wave functional of the free theory are evaluated and used to compute two-point functions. From the non-renormalization of the ghost-gluon vertex, the first coefficient of the $\beta$ function can be extracted.

**Article 3** shows an alternative approach to the Yang–Mills Schrödinger equation, based on a Volterra series for the exponent of the vacuum wave functional. The perturbative results of Article 2 are re-derived by a perturbative solution of the Schrödinger equation, and a diagrammatic technique to handle the occurring terms is developed. The one-loop perturbative results of Article 2 are further analysed, and the one-loop anomalous dimensions are evaluated through renormalization group improvement.

**Article 4** eventually focuses on a non-perturbative treatment of the vacuum wave functional beyond the Gaussian ansatz Eq. (I.128). After choosing a suitable ansatz, Dyson–Schwinger techniques are used to evaluate the vacuum expectation value of the Hamilton operator. The variation of the so-evaluated energy density with respect to the kernels of the ansatz allows to fix them. A new gap equation for the gluon propagator, which embodies a gluon-loop term, is derived.

**Article 5** presents the derivation of the Hamilton operator including the terms involving the vacuum angle $\theta$. The wave functional obtained previously in the $\theta = 0$ sector is exploited to evaluate the topological susceptibility, which is closely related to the mass of the $\eta'$ meson.

As a first step into the inclusion of dynamical fermions, the last chapter presents the perturbative treatment of quarks. The calculations are performed along the same lines of Article 2. The quark-loop contribution to the gluon propagator and its effect on the $\beta$ function are evaluated. Also the quark propagator and its one-loop renormalization are briefly discussed.

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Per omne fas et nefas.

*(Titus Livius)*

# Article 1

# Heat-kernel expansion and counterterms of the Faddeev–Popov determinant in Coulomb and Landau gauge

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**Abstract** The Faddeev–Popov determinant of Landau gauge in $d$ dimensions and Coulomb gauge in $d+1$ dimensions is calculated in the heat-kernel expansion up to next-to-leading order. The UV-divergent parts in $d = 3, 4$ are isolated and the counterterms required for a non-perturbative treatment of the Faddeev–Popov determinant are determined.

## 1.1 Introduction

In recent years there has been a strong interest in non-perturbative approaches to continuum Yang–Mills theory and much research has been carried out in this field. Most of these approaches rely on gauge fixing by the Faddeev–Popov method. This concerns, in particular, the Dyson–Schwinger equation approach in Landau gauge (for recent reviews see Refs. [1,2]) and in Coulomb gauge [3,4], and the Hamiltonian approach in Coulomb gauge [5,6]. Furthermore, in Coulomb and Landau gauge the Faddeev–Popov determinant is assumed to dominate the infrared sector of the theory, see e.g. Ref. [7]. For this reason, the spectrum of the Faddeev–Popov operator in Coulomb and Landau gauge has been subject to various lattice studies [8–11].

The standard way to renormalize the theory is to represent the Faddeev–Popov determinant by ghost fields and use the BRST invariance of the resulting local field theory. In some cases it is, however, advantageous not to introduce ghost fields but to keep the Faddeev–Popov determinant explicit and treat it non-perturbatively. This is, in particular, the case in the Hamiltonian approach in Coulomb gauge [6]. The renormalization of this theory requires then to identify the ultraviolet singular pieces of the Faddeev–Popov determinant and to remove them by appropriate counterterms. In Ref. [12] the UV coun-

terterms of the Faddeev–Popov determinant in $d = 3 + 1$ Coulomb gauge were identified using an approximate expression for the Faddeev-Popov determinant [13]

$$\ln \mathrm{Det}(-D\partial) = -\int \mathrm{d}^3x\, \mathrm{d}^3y\, A(x)\, \chi(x,y)\, A(y), \tag{1.1}$$

where $\chi$ is the so-called curvature, which represents the ghost loop contribution to the gluon self-energy. The representation (1.1) can be shown to be correct to 2-loop order for the energy. Not surprisingly the required counterterm to $\ln \mathrm{Det}(-D\partial)$ was found from this representation to be given by a mass-like term

$$C \int \mathrm{d}^3x\, A^2(x) \tag{1.2}$$

with the coefficient $C$ being linear in the UV cut-off. In the present paper we will calculate the counterterm to the Faddeev–Popov determinant exactly without resorting to the representation (1.1) and also determine the precise numerical coefficient of the counterterm. For this purpose we carry out a heat-kernel expansion of the Faddeev–Popov determinant in $d = 3, 4$ Landau gauge. The $d = 3$ Landau gauge corresponds to the $d = 3+1$ Coulomb gauge, the case of most interest to us.

The organization of the paper is as follows: in Sec. 1.2 we briefly summarize the heat-kernel expansion of functional determinants. In Sec. 1.3 we carry out the heat-kernel expansion for the Fadeev–Popov operator in Landau gauge. The corresponding heat coefficients are evaluated in Sec. 1.4, and the UV-divergent terms are isolated in Sec. 1.5. Finally in Sec. 1.6 we present our conclusions.

## 1.2 Heat-kernel expansion of functional determinants

We essentially follow Ref. [14] here. The determinant of a positive definite operator $M$ can be represented by the following proper-time integral

$$\mathrm{Tr} \ln M = -\int_{1/\Lambda^2}^{\infty} \mathrm{d}\tau\, \tau^{-1}\ \mathrm{Tr}\, K(\tau), \tag{1.3}$$

where $\Lambda$ is an ultraviolet momentum cut-off and

$$K(\tau) = \mathrm{e}^{-M\tau} \tag{1.4}$$

is the heat kernel. Furthermore, Tr includes both the trace over the discrete indices tr and the integration over space time $\int \mathrm{d}^dx$. The heat kernel (1.4) satisfies the heat equation

$$\partial_\tau K(\tau) + MK(\tau) = 0 \tag{1.5}$$

and the boundary condition

$$K(\tau = 0) = 1. \tag{1.6}$$

In the usual situation the operator under interest $M$ contains a "free" part $M_0$, which can be treated exactly. In this case it is convenient to express the full heat kernel as

$$\langle x|\, K(\tau)\, |y\rangle = \langle x|\, K_0(\tau)\, |y\rangle\ \langle x|\, H(\tau)\, |y\rangle, \tag{1.7}$$

where

$$K_0(\tau) = \mathrm{e}^{-\tau M_0} \tag{1.8}$$

is the free heat kernel satisfying

$$\big(\partial_\tau + M_0\big) K_0(\tau) = 0, \quad K_0(\tau = 0) = 1, \tag{1.9}$$

and $H(\tau)$ embodies all the "interactions". From the boundary conditions to the heat kernels follows the boundary condition for the interaction part

$$H(\tau = 0) = 1. \tag{1.10}$$

The interaction part $H(\tau)$ is then expanded in powers of the proper time $\tau$

$$\langle x|\, H(\tau)\, |y\rangle = \sum_{n=0}^{\infty} h_n(x,y)\, \tau^n, \tag{1.11}$$

where the $h_n(x,y)$ are referred to as heat coefficients. Equation (1.10) implies

$$h_0(x,x) = 1. \tag{1.12}$$

With Eqs. (1.7) and (1.11) we find for the functional determinant Eq. (1.3)

$$\mathrm{Tr} \ln M = -\sum_{n=0}^{\infty} \int_{1/\Lambda^2}^{\infty} \mathrm{d}\tau\, \tau^{n-1}\ \mathrm{Tr}\big(K_0(\tau) h_n\big). \tag{1.13}$$

The advantage of the heat-kernel expansion Eq. (1.13) is that with increasing order $n$ the terms become less and less ultraviolet singular and only the first few terms are ultraviolet singular while, starting at a certain $n$ (which depends on the number of space-time dimensions), the terms are ultraviolet finite. In the following we shall apply this heat-kernel expansion to the Faddeev–Popov determinant in Landau gauge in $d = 3$, 4 Euclidean dimensions.

## 1.3 Heat kernel evaluation of the Faddeev–Popov determinant in Landau gauge

The Faddev–Popov kernel in Landau gauge

$$\partial A = 0 \tag{1.14}$$

is given by

$$M = -D\partial, \tag{1.15}$$

where

$$D = \partial + A, \qquad A = A^a T^a, \qquad (T^a)^{bc} = f^{bac} \tag{1.16}$$

denotes the covariant derivative in the adjoint representation of the gauge field. Here and in the following all matrix valued quantities will be defined in the adjoint representation of the gauge group. Furthermore, we have absorbed the coupling constant in the gauge field. Note that the Faddeev–Popov operator (1.15) is positive definite in the first Gribov

region, to which the gauge fields should be restricted, and hence its determinant can be expressed by the proper-time integral Eq. (1.3). One should mention, however, that the Faddeev–Popov operator has constant zero modes corresponding to the global gauge invariance, which is not fixed by the Landau gauge condition (1.14). However, these zero modes occur also for the "unperturbed" operator $M_0 = -\partial^2$ and should hence cancel in the ratio $\mathrm{Det}\, M / \mathrm{Det}\, M_0$, which we will consider here.

To illustrate the troubles of the Gribov region in the continuum theory, consider constant gauge fields in $SU(2)$. For constant gauge fields the eigenvalues of the Faddeev–Popov operator are given by (see Ref. [15])

$$\lambda_{\mathbf{k},\sigma} = \mathbf{k}^2 - \sigma\, b\, |\mathbf{k}|, \qquad \sigma = 0, \pm 1 \tag{1.17}$$

where

$$b = \sqrt{(A_i^a \hat{k}_i)(A_j^a \hat{k}_j)} \qquad \hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|. \tag{1.18}$$

It is seen that even for very small $A_i^a = \text{const}$ there exists always momenta $|\mathbf{k}| < b$ so that the eigenvalue of Eq. (1.17) with $\sigma = 1$ becomes negative and the constant field configuration is outside the first Gribov region. This shows that in the continuum theory even arbitrary small (e.g. constant) gauge fields are outside the first Gribov region. On the lattice, constant gauge fields are inside the first Gribov region as long as $b$ [Eq. (1.18)] is smaller than the smallest non-zero momentum (the $\mathbf{k} = 0$ eigenvalue cancels against the $\mathbf{k} = 0$ eigenvalue of $M_0 = -\partial^2$). To guarantee that the sufficiently small gauge fields are inside the first Gribov region we will introduce an IR cut-off $\mu$ into the proper time integral by inserting the factor $\exp(-\mu^2\tau)$ into Eq. (1.13).

Choosing the unperturbed part of the Faddeev–Popov operator as

$$M_0 = -\partial^2, \tag{1.19}$$

the free heat kernel (1.8) in $d$-Euclidean dimensions is given by

$$\langle x|\, K_0(\tau)\, |y\rangle = (4\pi\tau)^{-d/2}\, \mathrm{e}^{-\frac{(x-y)^2}{4\tau}}. \tag{1.20}$$

Once the unperturbed part $M_0$ is specified, a recursion relation for the heat coefficients is derived from the heat equations (1.5), (1.9). With the above choice of $M$ (1.15) and $M_0$ (1.19), inserting the ansatz Eq. (1.7) into the heat equation (1.5) and exploiting the unperturbed heat equation (1.9), one obtains the following differential equation for the interaction part

$$\left[\partial_\tau - \partial^2 + \frac{1}{\tau}(x-y)\partial + \frac{1}{2\tau}(x-y)A - A\partial\right] \langle x|\, H(\tau)\, |y\rangle = 0. \tag{1.21}$$

Using the Landau gauge condition (1.14), this equation can be expressed as

$$\left[\partial_\tau + \frac{1}{\tau}(x-y)D - \frac{1}{2\tau}(x-y)A - D\partial\right] \langle x|\, H(\tau)\, |y\rangle = 0. \tag{1.22}$$

Inserting here the expansion of the interaction kernel in terms of the proper time [Eq. (1.11)] one derives the following recursion relation for the heat coefficients $h_n(x,y)$

$$\left[(k+1) + (x-y)\big(D - \frac{1}{2}A\big)\right] h_{k+1}(x,y) = D\partial h_k(x,y), \tag{1.23}$$

with the initial condition

$$(x-y)\big(D-\frac{1}{2}A\big)h_0(x,y)=0. \tag{1.24}$$

Note that the equations (1.23) and (1.24) are independent of the number of dimensions so that the heat coefficients $h_k$ are the same in all dimensions. The dependence on the number of dimensions is entirely contained in the free heat kernel [Eq. (1.20)].

Due to the boundary condition (1.12) the term with $n=0$ yields the determinant of the unperturbed kernel (1.19). We therefore find with the explicit expression for the unperturbed kernel (1.20) from (1.13)

$$\ln\frac{\mathrm{Det}(-D\partial)}{\mathrm{Det}(-\partial^2)}=-(4\pi)^{-d/2}\sum_{n=1}^{\infty}\int_{1/\Lambda^2}^{\infty}\mathrm{d}\tau\,\mathrm{e}^{-\mu^2\tau}\,\tau^{n-1-d/2}\;\mathrm{Tr}\,h_n\,, \tag{1.25}$$

where

$$\mathrm{Tr}\,h_n\equiv\int\mathrm{d}^dx\;\mathrm{tr}\,h_n(x,x). \tag{1.26}$$

From the representation (1.25) it is seen that the terms with

$$n-\frac{d}{2}\le 0 \tag{1.27}$$

are ultraviolet divergent, i.e. for $n<\frac{d}{2}$ we obtain power divergences $\Lambda^{d-2n}$, while for $n=\frac{d}{2}$ we obtain a logarithmic UV-divergence. Furthermore for $\mu=0$ the terms with $n-\frac{d}{2}\ge 0$ are infrared divergent, which is a manifestation of the presence of gauge fields escaping the first Gribov region as discussed above. Keeping $\mu$ finite yields

$$\ln\frac{\mathrm{Det}(-D\partial)}{\mathrm{Det}(-\partial^2)}=-(4\pi)^{-d/2}\sum_{n=1}^{\infty}\mu^{d-2n}\,\Gamma\Big(n-\frac{d}{2},\frac{\mu^2}{\Lambda^2}\Big)\,\mathrm{Tr}\,h_n\,, \tag{1.28}$$

where

$$\Gamma(a,z)=\int_z^{\infty}\mathrm{d}\tau\,\tau^{a-1}\,\mathrm{e}^{-\tau} \tag{1.29}$$

is the incomplete Gamma function. In the UV-finite terms $n>\frac{d}{2}$ we can let $\Lambda\to\infty$. Then the $\mu$-dependence of these terms is given by

$$\mu^{d-2n}, \tag{1.30}$$

i.e. these terms are diverging for $\mu\to 0$. We are interested here only in the UV-divergent part of the Faddeev–Popov determinant in $d=3$, 4, which can be easily extracted. For this purpose it is sufficient to calculate the heat coefficients up to including $h_2(x,x)$, which we will do next.

## 1.4 Calculation of the heat coefficients

The heat-kernels of generalized Laplacian operators were studied by a number of authors, see e.g. Ref. [16]. Below we explicitly calculate the leading and next-to-leading order heat coefficients of the Faddeev–Popov operator (1.15). To this purpose, it is useful to introduce

$$\bar{D}=\partial+\frac{1}{2}\,A=D-\frac{1}{2}\,A, \tag{1.31}$$

which is essentially a covariant derivative with a rescaled field. The Faddeev–Popov operator can then be rewritten as

$$-D\partial = -(\bar{D} + \tfrac{1}{2}A)(\bar{D} - \tfrac{1}{2}A) = -\bar{D}^2 + \tfrac{1}{4}A^2, \tag{1.32}$$

where we have used the Landau gauge condition $\partial A = [\bar{D}, A] = 0$, and the recursion relation for the heat coefficients Eq. (1.23) and the initial condition Eq. (1.24) become

$$(x-y)\bar{D}h_0(x,y) = 0, \tag{1.33a}$$

$$\big[(k+1) + (x-y)\bar{D}\big]h_{k+1}(x,y) = \big[\bar{D}^2 - \tfrac{1}{4}A^2\big]h_k(x,y). \tag{1.33b}$$

Putting $k = 0$ in the recursion relation (1.33b) and taking the coincidence limit $y \to x$ yields

$$h_1(x,x) = \big[\bar{D}^2 h_0(x,y)\big]_{x=y} - \frac{1}{4}\,A^2. \tag{1.34}$$

Acting with $\bar{D}_\mu\bar{D}_\nu$ on the initial condition Eq. (1.33a) yields

$$\begin{aligned} 0 &= (\bar{D}_\mu\bar{D}_\nu + \bar{D}_\nu\bar{D}_\mu)h_0(x,y) + \mathcal{O}(x-y) \\ &= \big\{2\bar{D}_\mu\bar{D}_\nu + \big[\bar{D}_\nu, \bar{D}_\mu\big]\big\}h_0(x,y) + \mathcal{O}(x-y) \end{aligned} \tag{1.35}$$

In the coincidence limit this yields

$$\big[\bar{D}_\mu\bar{D}_\nu h_0(x,y)\big]_{x=y} = \frac{1}{2}\,\bar{F}_{\mu\nu}\,, \tag{1.36}$$

where $\bar{F}_{\mu\nu}$ is the field stregth tensor constructed with the covariant derivative $\bar{D}$ (i.e. with the rescaled gauge field $A/2$). This implies that the first term in Eq. (1.34) vanishes, and the first heat coefficient reads

$$h_1(x,x) = -\frac{1}{4}\,A^2. \tag{1.37}$$

With the generators $T^a$ in the adjoint representation satisfying $\operatorname{tr} T^a T^b = -N_\mathrm{c}\delta^{ab}$ we eventually obtain

$$\operatorname{Tr} h_1 = \frac{N_\mathrm{c}}{4}\int \mathrm{d}^d x\, A^a_\mu(x)\, A^a_\mu(x). \tag{1.38}$$

To find the second heat coefficient $h_2$ we put $k = 1$ in the recusion relation (1.33b), yielding in the coincidence limit

$$2\,h_2(x,x) = \left[\bar{D}^2 - \frac{1}{4}A^2\right] h_1(x,y)\big|_{x=y} = \big(\bar{D}^2 h_1(x,y)\big)_{x=y} + \frac{1}{16}(A^2)^2. \tag{1.39}$$

where we have used Eq. (1.37). To calculate the right-hand side of Eq. (1.39) we act with the operator $\bar{D}^2$ on the recursion relation (1.33b) with $k = 0$

$$\bar{D}^2\big[1 + (x-y)\bar{D}\big]h_1(x,y) = \big[3\bar{D}^2 + \mathcal{O}(x-y)\big]h_1(x,y) = \bar{D}^2\big[\bar{D}^2 - \tfrac{1}{4}A^2\big]h_0(x,y) \tag{1.40}$$

which yields from Eq. (1.39)

$$h_2(x,x) = \frac{1}{6}\big(\bar{D}^2\bar{D}^2 - \frac{1}{4}\bar{D}^2A^2\big)h_0(x,y)\big|_{x=y} + \frac{1}{32}(A^2)^2. \tag{1.41}$$

The second term in Eq. (1.41) can be rewritten as

$$\bar{D}^2 A^2 = \big[\bar{D}_\mu, \big[\bar{D}_\mu, A^2\big]\big] + 2\big[\bar{D}_\mu, A^2\big]\bar{D}_\mu + A^2\bar{D}^2. \tag{1.42}$$

The last two terms give a vanishing contribution when acting on $h_0$ in the coincidence limit (see above), while the first one, being a total commutator, gives no contribution when the trace in color space is taken. Thus the whole term can be discarded. Applying the operator $\bar{D}^2\bar{D}^2$ on the initial condition (1.33a) one gets

$$0 = \bar{D}^2\bar{D}^2\big[(x-y)\bar{D}h_0(x,y)\big] = 2(\bar{D}^2\bar{D}^2 + \bar{D}_\mu\bar{D}^2\bar{D}_\mu)h_0(x,y) + \mathcal{O}(x-y). \tag{1.43}$$

Using the identity

$$\bar{D}^2\bar{D}^2 - \bar{D}_\mu\bar{D}^2\bar{D}_\mu = \bar{F}_{\mu\nu}\bar{F}_{\mu\nu} + \big[\bar{D}_\mu, \bar{F}_{\mu\nu}\big]\bar{D}_\nu \tag{1.44}$$

applied to $h_0$ and adding this to Eq. (1.43) we get finally

$$2\bar{D}^2\bar{D}^2 h_0(x,y) = \big(\bar{F}_{\mu\nu}^2 + \big[\bar{D}_\mu, \bar{F}_{\mu\nu}\big]\bar{D}_\nu\big)h_0(x,y). \tag{1.45}$$

The commutator term in Eq. (1.45) gives no contribution in the coincidence limit, since $\bar{D}_\mu h_0(x,y) = \mathcal{O}(x-y)$ by the initial condition (1.33a). The second heat coefficient reads then

$$\operatorname{tr} h_2(x,x) = \frac{1}{12}\operatorname{tr}\bar{F}_{\mu\nu}\bar{F}_{\mu\nu} + \frac{1}{32}\operatorname{tr}(A^2)^2. \tag{1.46}$$

The first term in Eq. (1.46) can of course be expressed in terms of the actual field strength tensor $F_{\mu\nu}$, which is related to $\bar{F}_{\mu\nu}$ by

$$\bar{F}_{\mu\nu} = \frac{1}{2}F_{\mu\nu} - \frac{1}{4}\big[A_\mu, A_\nu\big]. \tag{1.47}$$

The heat coefficient (1.46) then becomes

$$\operatorname{Tr} h_2 = \frac{1}{48}\int \mathrm{d}^d x \operatorname{tr}\left[\left(F_{\mu\nu} - \frac{1}{2}\big[A_\mu, A_\nu\big]\right)^2 + \frac{3}{2}(A^2)^2\right]. \tag{1.48}$$

## 1.5 The counterterms for $d = 3+1$ Coulomb gauge and $d = 4$ Landau gauge

We are interested here in the divergent part of the Faddeev–Popov determinant in $d = 3$ and $d = 4$ dimensions. We start with the $d = 3$ dimensional case, which corresponds to Coulomb gauge in $d = 3{+}1$ and is of most interest to us. In this case the only UV-divergent term in (1.25) is $n = 1$. Since

$$\Gamma\big(-\tfrac{1}{2}, x\big) = 2\left[\frac{e^{-x}}{\sqrt{x}} - \Gamma\big(\tfrac{1}{2}, x\big)\right], \tag{1.49}$$

this term is IR-finite so that we can take the limit $\mu = 0$ yielding

$$\ln\frac{\operatorname{Det}(-D\partial)}{\operatorname{Det}(-\partial^2)}\bigg|_{\mathrm{div}} = -\frac{1}{(4\pi)^{3/2}}\,2\Lambda\ \operatorname{Tr} h_1\,. \tag{1.50}$$

Inserting the explicit expression for the heat-coefficient $h_1$ (1.38), we obtain for the divergent part of the Faddeev–Popov determinant

$$\ln \frac{\mathrm{Det}(-D\partial)}{\mathrm{Det}(-\partial^2)}\bigg|_{\mathrm{div}} = -\frac{N_\mathrm{c}}{2}\,(4\pi)^{-3/2}\Lambda \int \mathrm{d}^3x \big(A_i^a(x)\big)^2. \tag{1.51}$$

This result is consistent with Ref. [12], where it was found that (to the order considered) the renormalization of the Faddeev-Popov determinant requires a counterterm of the type (1.2). Here we have proven that, in general, in $d = 3$ there is only one UV-divergent counterterm required and that the coefficient of this term is linearly divergent.

In $d = 4$ the terms in Eq. (1.28) with $n = 1$, 2 are UV-divergent yielding

$$\ln \frac{\mathrm{Det}(-D\partial)}{\mathrm{Det}(-\partial^2)}\bigg|_{\mathrm{div}} = -\frac{1}{(4\pi)^2}\left[\Lambda^2\ \mathrm{Tr}\, h_1 + \ln\frac{\Lambda^2}{\mu^2}\ \mathrm{Tr}\, h_2\right]. \tag{1.52}$$

The heat coefficients $h_n$ are the same as in $d = 3$ and are given by Eqs. (1.37) and (1.48).

A comment is here in order: it is well known that gauge invariance forbids the occurrence of quadratically divergent mass terms. In fact, these terms disappear due to a cancellation between contributions from ghost and gluon loops. Since we are dealing here only with the Faddeev–Popov determinant, i.e. with the ghost loop, such terms will occur.

## 1.6 Summary and conclusions

We have performed a heat-kernel expansion of the Faddeev–Popov determinant in Landau/Coulomb gauge. We have shown that in $d = 3$ Landau ($d = 3 + 1$ Coulomb) gauge there is a single, unique UV-divergent counterterm required, which is given by Eq. (1.51). It represents a mass term whose coefficient is linearly UV-divergent. In $d = 4$ Landau gauge there are two UV counterterms required, which are respectively quadratically and logarithmically UV-divergent. The latter term and all the UV-finite terms are IR-singular, reflecting the absence of a mass scale in the original theory. However, as is well known the scale anomaly gives rise to dimensional transmutation which manifests itself in gluon condensation and the generation of a dynamical mass scale. To access the UV-finite terms in the heat-kernel expansion, this dynamical mass scale has to be included in the unperturbed heat-kernel. In principle, the heat-kernel expansion is a type of gradient expansion, which should be applicable in the infrared, provided the unperturbed kernel is adequately choosen. This expansion is, however, bound to fail near the Gribov horizon, where the Faddeev–Popov operator develops a zero eigenvalue.

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In die, perniciosum;
in hebdomada, utile;
in mense, necessarium.

*(Aulus Cornelius Celsus)*

# Article 2

# Perturbation theory in the Hamiltonian approach to Yang–Mills theory in Coulomb gauge

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**Abstract** We study the Hamiltonian approach to Yang–Mills theory in Coulomb gauge in Rayleigh–Schrödinger perturbation theory. The static gluon and ghost propagator as well as the potential between static colour sources are calculated to one-loop order. Furthermore, the one-loop $\beta$ function is calculated from both the ghost-gluon vertex and the static potential and found to agree with the result of covariant perturbation theory.

## 2.1 Introduction

In recent years, there has been a renewed interest in Yang–Mills theory in Coulomb gauge both in the continuum [1–8] and on the lattice [9–12]. This gauge has several advantages over the frequently used Landau gauge. Among these are: the use of the physical degrees of freedom (at least in QED) and the explicit emergence of a static colour charge potential. Furthermore, in this gauge the form factor of the ghost propagator represents the dielectric function of the Yang–Mills vacuum [13]. The disadvantage of this gauge is, of course, that it is non-covariant, which is considered a drawback for perturbation theory, and renormalisability is yet to be proven. Recently, there has been much activity in the Hamiltonian approach to Yang–Mills theory in Coulomb gauge [1,2,5,6,13–16]. By means of a physically motivated ansatz for the vacuum wave functional, a variational solution of the Yang–Mills Schrödinger equation has been accomplished [2,3,5]. The so-called gap equation resulting from the minimization of the vacuum energy density was converted

into a set of Dyson–Schwinger equations for the static gluon, ghost and Coulomb propagators. Due to the particular ansatz for the vacuum wave functional used, which grasps the essential infrared physics, the resulting gluon propagator does not yield the proper ultraviolet (UV) asymptotics known from perturbation theory. The reason is that the three-gluon vertex does not contribute to the vacuum energy density for the ansatz of the wave functional considered.

In the present paper, we carry out a thorough perturbative calculation in the Hamiltonian approach to Yang–Mills theory. We will use the standard Rayleigh–Schrödinger perturbation theory and carry out the calculations up to and including one-loop order. In particular, we will produce the correct one-loop expression for the perturbative $\beta$ function which has been found in the functional integral approach [7,8] before. Because of asymptotic freedom, we thus have a correct description of the UV regime (to one-loop precision). With the correct perturbative results at hand, we show how the variational approach has to be modified to yield the correct UV asymptotics. Furthermore, the perturbative calculations performed in the present paper can be extended to calculate systematic corrections to the non-perturbative variational vacuum solution and to construct a complete basis for the Hilbert space of Yang–Mills theory, which is required for the calculation of the partition function or free energy. In a forthcoming paper, the latter will be minimized to achieve a description of the deconfinement phase transition at finite temperatures in the Hamiltonian approach.

## 2.2 Perturbative expansion of the Yang–Mills Hamiltonian

### 2.2.1 The Yang–Mills Hamiltonian in Coulomb gauge

In the absence of matter fields, the Yang–Mills Hamiltonian in Coulomb gauge reads [17]

$$H_{\text{YM}} = \int \mathrm{d}^d x \left[ \frac{1}{2}\, \mathcal{J}_A^{-1}\, \Pi_i^a(\mathbf{x})\, \mathcal{J}_A\, \Pi_i^a(\mathbf{x}) + \frac{1}{4}\, F_{ij}^a(\mathbf{x})\, F_{ij}^a(\mathbf{x}) \right] + \\ + \frac{g^2}{2} \int \mathrm{d}^d x\, \mathrm{d}^d y\, \hat{A}_i^{ac}(\mathbf{x})\, \mathcal{J}_A^{-1}\, \Pi_i^c(\mathbf{x})\, \mathcal{J}_A\, F_A^{ab}(\mathbf{x},\mathbf{y}) \hat{A}_j^{bd}(\mathbf{y}) \Pi_j^d(\mathbf{y}), \quad (2.1)$$

where $\mathbf{x}$ is a vector in $d$ space dimensions, $A_i^a$ is the transverse gauge field operator, $\Pi_i^a$ is the transverse momentum operator satisfying the canonical commutation relations

$$\left[A_i^a(\mathbf{x}), \Pi_j^b(\mathbf{y})\right] = \mathrm{i}\, \delta^{ab}\, t_{ij}(\mathbf{x})\, \delta(\mathbf{x}-\mathbf{y}), \qquad t_{ij}(\mathbf{x}) = \delta_{ij} - \partial_i \partial_j / \partial^2, \quad (2.2)$$

and

$$F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + g\, f^{abc} A_i^b A_j^c \quad (2.3)$$

is the field strength tensor ($g$ is the coupling constant, $f^{abc}$ are the structure constants of the $\mathfrak{su}(N_\mathrm{c})$ algebra, and $\hat{A}^{ac} = f^{abc} A^b$ is the gauge field operator in the adjoint colour representation). Furthermore,

$$\mathcal{J}_A = \mathrm{Det}(-\hat{D}\cdot\partial)/\,\mathrm{Det}(-\partial^2), \qquad \hat{D}_i^{ab} = \delta^{ab}\,\partial_i + g\hat{A}_i^{ab} \quad (2.4)$$

is the Faddeev–Popov determinant of Coulomb gauge, and

$$F_A^{ab}(\mathbf{x},\mathbf{y}) = \left[ (-\hat{D}\cdot\partial)^{-1}\, (-\partial^2)\, (-\hat{D}\cdot\partial)^{-1} \right]_{\mathbf{x},\mathbf{y}}^{ab} \quad (2.5)$$

the so-called Coulomb kernel.

The Coulomb gauge $\partial_i A_i^a = 0$ is implemented in the scalar product of the Hilbert space of Yang–Mills wave functionals by the Faddeev–Popov method, yielding for the matrix elements of observables $O[A, \Pi]$

$$\langle \psi_1 | \, O \, | \psi_2 \rangle = \int \mathcal{D}A \, \mathcal{J}_A \, \psi_1^*[A] \, O[A, \Pi] \, \psi_2[A], \tag{2.6}$$

where the integration is over the transverse gauge fields only.

It is convenient to remove the Jacobian $\mathcal{J}_A$ from the integration measure by defining

$$\psi[A] = \mathcal{J}_A^{-1/2} \, \widetilde{\psi}[A], \qquad \widetilde{O} = \mathcal{J}_A^{1/2} O \mathcal{J}_A^{-1/2}. \tag{2.7}$$

The transformed Hamiltonian $\widetilde{H}$ is then obtained from Eq. (2.1) by replacing the momentum operator $\Pi$ by the transformed one

$$\widetilde{\Pi}_i^a(\mathbf{x}) = \mathcal{J}_A^{1/2} \, \Pi_i^a(\mathbf{x}) \mathcal{J}_A^{-1/2} = \Pi_i^a(\mathbf{x}) + \frac{\mathrm{i}}{2} \frac{\delta \ln \mathcal{J}_A}{\delta A_i^a(\mathbf{x})} \, . \tag{2.8}$$

This yields

$$\begin{aligned} \widetilde{H}_{\mathrm{YM}} = & \int \mathrm{d}^d x \left[ \frac{1}{2} \widetilde{\Pi}_i^{a\dagger}(\mathbf{x}) \, \widetilde{\Pi}_i^a(\mathbf{x}) + \frac{1}{4} F_{ij}^a(\mathbf{x}) \, F_{ij}^a(\mathbf{x}) \right] + \\ & + \frac{g^2}{2} \int \mathrm{d}^d x \, \mathrm{d}^d y \, \hat{A}_i^{ac}(\mathbf{x}) \widetilde{\Pi}_i^{c\dagger}(\mathbf{x}) F_A^{ab}(\mathbf{x}, \mathbf{y}) \hat{A}_j^{bd}(\mathbf{y}) \widetilde{\Pi}_j^d(\mathbf{y}). \end{aligned} \tag{2.9}$$

In the following, Dirac's bra-ket notation will refer to the transformed space where no Faddeev–Popov determinant occurs in the functional integration measure.

For perturbation theory, it will be convenient to expand the gauge field in Fourier modes. We use the following conventions

$$A_i^a(\mathbf{x}) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \, A_i^a(\mathbf{p}), \quad \Pi_i^a(\mathbf{x}) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \, \Pi_i^a(\mathbf{p}), \tag{2.10}$$

where the transformed fields satisfy the canonical commutations relations

$$\left[ A_i^a(\mathbf{k}), \Pi_j^b(\mathbf{p}) \right] = \mathrm{i} \, \delta^{ab} t_{ij}(\mathbf{k}) \, (2\pi)^d \, \delta(\mathbf{k} + \mathbf{p}), \tag{2.11}$$

and where $t_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / \mathbf{k}^2$ is the transverse projector in momentum space. To simplify the notation, we introduce the following shortcuts

$$đp \equiv \frac{\mathrm{d}^d p}{(2\pi)^d} \, , \qquad đ(\mathbf{p}) \equiv (2\pi)^d \delta(\mathbf{p}). \tag{2.12}$$

### 2.2.2 Expansion of the Hamiltonian

Expanding the Hamiltonian $\widetilde{H}_{\mathrm{YM}}$ [Eq. (2.9)] in powers of the coupling constant $g$, thereby using $\mathcal{J}_A = 1 + \mathcal{O}(g^2)$ and

$$\widetilde{\Pi}^\dagger \widetilde{\Pi} = \Pi^2 - \mathcal{J}_A^{-1/2} \big[ \Pi, \big[ \Pi, \mathcal{J}_A^{1/2} \big] \big], \tag{2.13}$$

we obtain

$$\widetilde{H} = H_0 + g\widetilde{H}_1 + g^2\widetilde{H}_2 + \mathcal{O}(g^3), \tag{2.14}$$

where the unperturbed Hamiltonian

$$H_0 = \frac{1}{2}\int \mathrm{d}^d x \left[\left(\Pi_i^a(\mathbf{x})\right)^2 - A_i^a(\mathbf{x})\,\partial^2 A_i^a(\mathbf{x})\right] \tag{2.15}$$

is the Hamiltonian of QED except for the extra colour index of the gauge field. The first order term $g\widetilde{H}_1$ arises from the expansion of the magnetic energy $\int \mathrm{d}^d x F_{ij}^2$ and is given by the three-gluon vertex

$$\widetilde{H}_1 = \frac{\mathrm{i}}{3!}\, f^{a_1a_2a_3}\int đk_1\; đk_2\; đk_3\, \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\, T(1,2,3)\, A(1)\, A(2)\, A(3). \tag{2.16}$$

Here we have introduced the shorthand notation $A(1) \equiv A_{i_1}^{a_1}(\mathbf{k}_1)$ and $T(1,2,3)$ carries the (totally antisymmetric) Lorentz structure of the three-gluon vertex,

$$T(1,2,3) := t_{i_1 j}(\mathbf{k}_1)\, t_{i_2 l}(\mathbf{k}_2)\, t_{i_3 m}(\mathbf{k}_3)\big[\delta_{jl}(k_2-k_1)_m + \delta_{lm}(k_3-k_2)_j + \delta_{jm}(k_1-k_3)_l\big]. \tag{2.17}$$

Finally, the second-order term

$$\begin{aligned}\widetilde{H}_2 = C + \frac{1}{2}\, f^{aa_1a_2} f^{aa_3a_4}\int đk_1 \ldots đk_4\, \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4) \\ \times\left[\frac{\delta_{i_1i_3}\,\delta_{i_2i_4}}{2}\, A(1)\, A(2)\, A(3)\, A(4) + \frac{\delta_{i_1i_2}\,\delta_{i_3i_4}}{(\mathbf{k}_1+\mathbf{k}_2)^2}\, A(1)\,\Pi(2)\, A(3)\,\Pi(4)\right].\end{aligned} \tag{2.18}$$

contains besides the usual four-gluon vertex (first term in the bracket) also a contribution from the Coulomb term (second term in the bracket) arising from the expansion of the Coulomb kernel (2.5). Note that the Coulomb term is already $\mathcal{O}(g^2)$, see Eq. (2.9), so that to the order considered we can replace the Coulomb kernel Eq. (2.5) simply by its bare form $(-\partial^2)^{-1}$. The first term in Eq. (2.18) is an irrelevant constant arising from the expansion of the second term in Eq. (2.13). Since such a constant does not influence the wave functional, we will skip it in the following.

Since we will use dimensional regularization, in order to preserve the dimension of the dressing functions we will replace

$$g \to g\mu^{(3-d)/2}, \tag{2.19}$$

with $\mu$ being an arbitrary mass scale.

### 2.2.3 The unperturbed basis

The perturbative vacuum state $\psi_0[A] = \langle A|0\rangle$ is given by

$$\langle A|0\rangle = \mathcal{N}\exp\left\{-\frac{1}{2}\int đk\; A_i^a(\mathbf{k})\, t_{ij}(\mathbf{k})\,|\mathbf{k}|\, A_j^a(-\mathbf{k})\right\}. \tag{2.20}$$

Up to the colour index of the gauge field, this is precisely the exact vacuum wave functional for QED without fermions. This state is the lowest energy eigenstate of $H_0$ [Eq. (2.15)] with energy

$$E_0 = (N_c^2-1)\frac{d-1}{2}\,\delta(\mathbf{0})\int đk\,|\mathbf{k}|, \tag{2.21}$$

where $\delta(\mathbf{0}) = \int \mathrm{d}^d x$ is the spatial volume. To restrict the Coulomb gauge field to its transverse degrees of freedom, it is convenient to introduce the eigenvectors of the transverse projector $t_{ij}(\mathbf{k})$ in $d$ spatial dimensions with eigenvalue 1

$$t_{ij}(\mathbf{k})\, c_j^\sigma(\mathbf{k}) = c_i^\sigma(\mathbf{k}), \tag{2.22}$$

where $\sigma$ labels the $d-1$ different eigenvectors. From $k_i t_{ij}(\mathbf{k}) = 0$ it follows immediately that these eigenvectors are orthogonal to $\mathbf{k}$

$$k_i c_i^\sigma(\mathbf{k}) = 0. \tag{2.23}$$

Assuming the normalization

$$c_i^{\sigma *}(\mathbf{k})\, c_i^\tau(\mathbf{k}) = \delta_{\sigma\tau}\,, \tag{2.24}$$

these vectors satisfy the "completeness" relation in the transverse subspace

$$c_i^\sigma(\mathbf{k})\, c_j^{\sigma^*}(\mathbf{k}) = t_{ij}(\mathbf{k}). \tag{2.25}$$

Since the transverse projector $t_{ij}(\mathbf{k})$ is a real symmetric matrix, it can be diagonalized by an orthogonal transformation and hence the eigenvectors $c_i^\sigma(\mathbf{k})$ can be chosen real, which we will assume below. Let us mention, however, that in $d = 3$ the $c_i^\sigma(\mathbf{k})$ are usually chosen as the circular polarization vectors, which are complex.

The transverse components of the gauge field and their momenta are then given by

$$A_\sigma^a(\mathbf{k}) := c_i^\sigma(\mathbf{k}) A_i^a(\mathbf{k}), \qquad \Pi_\sigma^a(\mathbf{k}) := c_i^\sigma(\mathbf{k}) \Pi_i^a(\mathbf{k}). \tag{2.26}$$

In view of Eq. (2.11) they satisfy the commutation relation

$$\left[A_\sigma^a(\mathbf{k}), \Pi_\tau^b(\mathbf{p})\right] = \mathrm{i}\, \delta_{\sigma\tau} \delta^{ab} \delta\!\!\!^{-}(\mathbf{k}+\mathbf{p}). \tag{2.27}$$

In the transverse components of the gauge field, the perturbative vacuum state (2.20) reads

$$\langle A|0\rangle = \mathcal{N} \exp\left\{-\frac{1}{2}\int \mathrm{đ}k\; A_\sigma^a(\mathbf{k})\, |\mathbf{k}|\, A_\sigma^a(-\mathbf{k})\right\}. \tag{2.28}$$

This state is annihilated by the operator

$$a_\sigma^a(\mathbf{k}) = \sqrt{\frac{|\mathbf{k}|}{2}}\left[A_\sigma^a(\mathbf{k}) + \frac{\mathrm{i}}{|\mathbf{k}|}\,\Pi_\sigma^a(\mathbf{k})\right], \qquad a_\sigma^a(\mathbf{k})\,|0\rangle = 0, \tag{2.29}$$

which together with its Hermitian conjugate

$$a_\sigma^{a\dagger}(\mathbf{k}) = \sqrt{\frac{|\mathbf{k}|}{2}}\left[A_\sigma^a(-\mathbf{k}) - \frac{\mathrm{i}}{|\mathbf{k}|}\,\Pi_\sigma^a(-\mathbf{k})\right] \tag{2.30}$$

fulfills the usual Bose commutation relation

$$\left[a_\sigma^a(\mathbf{k}), a_\tau^{b\dagger}(\mathbf{p})\right] = \delta^{ab}\delta_{\sigma\tau}\, \delta\!\!\!^{-}(\mathbf{k}-\mathbf{p}). \tag{2.31}$$

The unperturbed Hamiltonian Eq. (2.15) is diagonalized by the transformations (2.29), (2.30)

$$H_0 = E_0 + \int \mathrm{đ}k\; |\mathbf{k}|\, a_\sigma^{a\dagger}(\mathbf{k})\, a_\sigma^a(\mathbf{k}), \tag{2.32}$$

and accordingly the eigenfunctions of $H_0$ are the multiple gluon states defined by

$$|g_1 g_2 \dots g_N\rangle = a^{a_1\dagger}_{\sigma_1}(\mathbf{k}_1)\, a^{a_2\dagger}_{\sigma_2}(\mathbf{k}_2) \dots a^{a_N\dagger}_{\sigma_N}(\mathbf{k}_N)\,|0\rangle \tag{2.33}$$

with energies

$$E_0 + \sum_{i=1}^{N} |\mathbf{k}_i|. \tag{2.34}$$

There are two ways to proceed now: given the unperturbed basis (2.33), one can express the gauge field $A$ and its momentum operator $\Pi$ in the perturbations, Eqs. (2.16) and (2.18), in terms of the creation and annihilation operators $a^\dagger_\sigma$, $a_\sigma$. Alternatively, since the perturbation $\widetilde{H}_{1,2}$ is expressed in terms of the gauge field and its momentum, one may wish to express the unperturbed basis states through $A$ and $\Pi$ using Eq. (2.30). This yields

$$|g\rangle = \sqrt{2|\mathbf{k}|}\, A^a_\sigma(-\mathbf{k})\,|0\rangle. \tag{2.35}$$

for a one-particle state, and

$$|g_1, g_2\rangle = \left[2\sqrt{|\mathbf{k}_1|\,|\mathbf{k}_2|}\; A^{a_1}_{\sigma_1}(-\mathbf{k}_1)\, A^{a_2}_{\sigma_2}(-\mathbf{k}_2) - \delta^{a_1 a_2}\,\delta_{\sigma_1\sigma_2}\bar{\delta}(\mathbf{k}_1+\mathbf{k}_2)\right]|0\rangle, \tag{2.36}$$

for a two-particle state, where the second term on the right-hand side of the last equation arises from the canonical commutation relations. States with more gluons have similar additional contraction terms. These contact terms ensure that the free $n$-gluon states are orthogonal to each other and to the unperturbed vacuum. It turns out that the contact terms simply eliminate from the matrix elements of observables the contractions of gauge field operators stemming exclusively from the wave functionals. We can therefore use the simplified representation

$$|g_1, \dots, g_n\rangle = \left[\prod_{i=1}^{n} \sqrt{2|\mathbf{k}_i|}\; A^{a_i}_{\sigma_i}(-\mathbf{k}_i)\right]|0\rangle, \tag{2.37}$$

with the additional calculational rule that in the evaluation of matrix elements of the form $\langle 0|\, O[A,\Pi]\,|g_1,\dots,g_n\rangle$, the $A$ fields in the states Eq. (2.37) must not be contracted with each other but only with $A$, $\Pi$ occurring in the observable $O[A,\Pi]$ .

### 2.2.4 Expansion of the vacuum wave functional

In Rayleigh–Schrödinger perturbation theory, the leading order (in the coupling constant) corrections to the vacuum wave functional are given by

$$|0\rangle^{(1)} = -\sum_n \frac{1}{n!}\,\frac{\langle g_1 \dots g_n|\,\widetilde{H}_1\,|0\rangle}{|\mathbf{k}_1| + \dots + |\mathbf{k}_n|}\;|g_1 \dots g_n\rangle, \tag{2.38a}$$

$$|0\rangle^{(2)} = -\sum_n \frac{1}{n!}\,\frac{1}{|\mathbf{k}_1| + \dots + |\mathbf{k}_n|}\left[\langle g_1 \dots g_n|\,\widetilde{H}_2\,|0\rangle + \langle g_1 \dots g_n|\,\widetilde{H}_1\,|0\rangle^{(1)}\right]|g_1 \dots g_n\rangle, \tag{2.38b}$$

where $\widetilde{H}_1$ and $\widetilde{H}_2$ are defined in Eqs. (2.16), (2.18), and the factors $1/n!$ avoid multiple counting due to identical gluons (summation over colour and polarization indices and integration over the momenta is implicit).

It is now straightforward to calculate the perturbative corrections to the vacuum wave functional. Since $\tilde{H}_1$ contains three field operators and is antisymmetric in both colour and Lorentz indices, in view of Eq. (2.37) it is clear that only three-gluon states will contribute to $|0\rangle^{(1)}$ [Eq. (2.38a)], yielding

$$|0\rangle^{(1)} = \frac{\mathrm{i}}{3!} f^{a_1 a_2 a_3} \int \mathrm{đ}k_1\, \mathrm{đ}k_2\, \mathrm{đ}k_3 \frac{\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)}{|\mathbf{k}_1|+|\mathbf{k}_2|+|\mathbf{k}_3|} T(1,2,3)\, A(-1)\, A(-2)\, A(-3)\, |0\rangle. \quad (2.39)$$

As a consequence, the second term in $|0\rangle^{(2)}$ [Eq. (2.38b)] receives contributions from up to six-gluon states. Furthermore, since $\widetilde{H}_2$ contains terms with up to four field operators, the first term in $|0\rangle^{(2)}$ will receive contributions from two- and four-gluon states. However, it turns out that up to order $g^2$ only two-gluon states contribute to the static propagators, so that

$$|0\rangle^{(2)} = -N_\mathrm{c}\, \frac{\delta^{a_1 a_2}}{8} \int \mathrm{đ}k_1\, \mathrm{đ}k_2 \frac{\delta(\mathbf{k}_1+\mathbf{k}_2)}{|\mathbf{k}_1|} \left[F_B(1,2)+F_C(1,2)+F_1(1,2)\right] A(-1)\, A(-2)\, |0\rangle, \quad (2.40)$$

where we have introduced the abbreviations

$$F_B(1,2) = \frac{1}{2} \int \mathrm{đ}q\, \frac{(d-1)t_{i_1 i_2}(\mathbf{k}_1) - t_{i_1 i_2}(\mathbf{q})}{|\mathbf{q}|}, \quad (2.41a)$$

$$F_C(1,2) = \int \mathrm{đ}q\, \frac{t_{i_1 i_2}(\mathbf{q})}{(\mathbf{k}_1-\mathbf{q})^2} \left[|\mathbf{q}| - \frac{\mathbf{k}_1^2}{|\mathbf{q}|}\right], \quad (2.41b)$$

$$F_1(1,2) = -\frac{1}{2} \int \mathrm{đ}k_3\; \mathrm{đ}k_4\, \frac{T(-1,3,4)\, T(2,3,4)}{|\mathbf{k}_1|+|\mathbf{k}_3|+|\mathbf{k}_4|} \frac{\delta(\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)}{|\mathbf{k}_3|\,|\mathbf{k}_4|}. \quad (2.41c)$$

Here, $F_B(1,2)$ arises from the four-gluon vertex, $F_C(1,2)$ from the Coulomb term, and $F_1(1,2)$ from the second-order contribution of the three-gluon vertex, see the last term in Eq. (2.38b). We have also used the shorthand notation $A(-1) = A^{a_1}_{i_1}(-\mathbf{k}_1)$, etc. In the Rayleigh–Schrödinger perturbation theory used in Eqs. (2.38), the correction to a wave function of a given order is chosen to be orthogonal to the unperturbed wave function,

$$\langle 0|0\rangle^{(i)} = 0, \qquad i \geq 1. \quad (2.42)$$

As a consequence, the wave functions obtained in a given order of perturbation theory are not properly normalized. We are interested in the vacuum wave functional up to second order

$$|\Omega\rangle = \mathcal{N}_\Omega \left[|0\rangle + g\,|0\rangle^{(1)} + g^2\,|0\rangle^{(2)} + \mathcal{O}(g^3)\right]. \quad (2.43)$$

Calculating the normalization constant $\mathcal{N}_\Omega$ up to order $g^2$ making use of Eq. (2.42), we find for the properly normalized vacuum wave functional up to this order

$$|\Omega\rangle = \left[1 - \frac{g^2}{2}\, {}^{(1)}\langle 0|0\rangle^{(1)}\right] |0\rangle + g\,|0\rangle^{(1)} + g^2\,|0\rangle^{(2)} + \mathcal{O}(g^3). \quad (2.44)$$

In the following sections we will use this perturbative expansion of the vacuum wave functional to calculate various static propagators.

## 2.3 Ghost propagator

The Green function (or inverse) $G_A$ of the Faddeev–Popov operator is defined by

$$-\hat{D}_i^{ab}\partial_i^x\, G_A^{bc}(\mathbf{x},\mathbf{y}) = \delta^{ac}\,\delta(\mathbf{x}-\mathbf{y}). \tag{2.45}$$

Expanding $G_A$ in a power series in the coupling constant

$$G_A^{ab}(\mathbf{x},\mathbf{y}) = \sum_{n=0} g^n\, G_n^{ab}[A](\mathbf{x},\mathbf{y}) \tag{2.46}$$

we get from (2.45) the recursion relation

$$-\partial_x^2\, G_{n+1}^{ab}[A](\mathbf{x},\mathbf{y}) = \hat{A}_i^{ac}(\mathbf{x})\partial_i^x\, G_n^{cb}[A](\mathbf{x},\mathbf{y}), \qquad n \geq 0 \tag{2.47a}$$

together with the initial condition

$$-\partial_x^2\, G_0^{ab}[A](\mathbf{x},\mathbf{y}) = \delta^{ab}\,\delta(\mathbf{x}-\mathbf{y}), \tag{2.47b}$$

which defines the unperturbed static ghost propagator $G_0^{ab}[A](\mathbf{x},\mathbf{y})$ as the Green function of the Laplacian, which in momentum space reads

$$G_0^{ab}[A](\mathbf{k}) = \delta^{ab}\, G_0(\mathbf{k}) = \frac{\delta^{ab}}{\mathbf{k}^2}\,. \tag{2.48}$$

It is diagonal in colour space and independent of the gauge field. With this property, it follows from Eq. (2.47a) that each term $G_n[A]$ contains a product of $n$ gauge field operators.

The ghost propagator is defined as the expectation value of the inverse Faddeev–Popov operator in the vacuum state $|\Omega\rangle$,

$$G^{ab}(\mathbf{x},\mathbf{y}) = \langle\Omega|\, G_A^{ab}(\mathbf{x},\mathbf{y})\,|\Omega\rangle. \tag{2.49}$$

Since $G_A$ does not depend on the momentum operator $\Pi$, it does not change under the transformation to the "radial" Hilbert space, i.e. $\widetilde{G}_A = G_A$ [see Eq. (2.7)]. Furthermore, contrary to the Faddeev–Popov operator, the vacuum expectation value of its inverse, the static ghost propagator, is translationally invariant.

Inserting the expansions Eq. (2.46) for $G_A$ and Eq. (2.44) for $|\Omega\rangle$ into Eq. (2.49) it is possible to show that many terms vanish or cancel, so that the ghost propagator reduces to

$$G^{ab}(\mathbf{x},\mathbf{y}) = G_0^{ab}(\mathbf{x},\mathbf{y}) + g^2\langle 0|\, G_2^{ab}[A](\mathbf{x},\mathbf{y})\,|0\rangle + \mathcal{O}(g^3). \tag{2.50}$$

The second-order term in Eq. (2.50) can be evaluated by means of the recursion relation Eq. (2.47a), yielding

$$G_2^{ab}[A](\mathbf{x},\mathbf{y}) = \Big[G_0\,(\hat{A}\cdot\partial)\,G_0\,(\hat{A}\cdot\partial)\,G_0\Big]_{\mathbf{x},\mathbf{y}}^{ab}\,, \qquad G_0 = (-\partial^2)^{-1}. \tag{2.51}$$

The vacuum expectation value of Eq. (2.51) can be expressed through the bare static gluon propagator, which in view of Eq. (2.20) reads

$$D_0(1,2) = \langle 0|\, A(1)\, A(2)\,|0\rangle = \delta^{a_1 a_2}\,\delta(\mathbf{k}_1+\mathbf{k}_2)\,\frac{t_{i_1 i_2}(\mathbf{k}_1)}{2|\mathbf{k}_1|}\,. \tag{2.52}$$

Then for the ghost form factor $D_c(\mathbf{k})$ defined by

$$G(\mathbf{x},\mathbf{y}) =: \int \text{đ}k\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\, \frac{D_c(\mathbf{k})}{\mathbf{k}^2}\,, \tag{2.53}$$

we get the following expression at one-loop order

$$D_c(\mathbf{k}) = 1 + g^2\,\mu^{3-d}\,\frac{N_\mathrm{c}}{2\mathbf{k}^2}\int \text{đ}q\,\frac{k_i\,k_j\,t_{ij}(\mathbf{q})}{(\mathbf{k}-\mathbf{q})^2\,|\mathbf{q}|} + \mathcal{O}(g^3). \tag{2.54}$$

The integral (2.54) is standard and can be evaluated in dimensional regularization with $d = 3 - 2\varepsilon$ in the usual way, yielding

$$D_c(\mathbf{k}) = 1 + g^2\,\frac{N_\mathrm{c}}{(4\pi)^{2-\varepsilon}}\left[\frac{4}{3}\left(\frac{1}{\varepsilon} - \ln\frac{\mathbf{k}^2}{\mu^2} - \gamma_\mathrm{E}\right) - \frac{8}{3}\,\ln 2 + \frac{28}{9} + \mathcal{O}(\varepsilon)\right]. \tag{2.55}$$

## 2.4 Gluon propagator

### 2.4.1 Gluon propagator in the Hamiltonian approach

The full static gluon propagator is defined in momentum space by

$$\tilde{\delta}(\mathbf{k}+\mathbf{p})\,D^{ab}_{ij}(\mathbf{k}) = \langle\Omega|\,A^a_i(\mathbf{k})\,A^b_j(\mathbf{p})\,|\Omega\rangle. \tag{2.56}$$

With the expansion (2.44) for the vacuum functional, the nonvanishing terms up to order $\mathcal{O}(g^2)$ are

$$\begin{aligned}\tilde{\delta}(\mathbf{k}_1+\mathbf{k}_2)\,D^{a_1a_2}_{i_1i_2}(\mathbf{k}_1) = D_0(1,2)\left[1 - g^2\,{}^{(1)}\!\langle 0|0\rangle^{(1)}\right] + \\ + g^2\left[{}^{(1)}\!\langle 0|\,A(1)\,A(2)\,|0\rangle^{(1)} + \langle 0|\,A(1)\,A(2)\,|0\rangle^{(2)} + {}^{(2)}\!\langle 0|\,A(1)\,A(2)\,|0\rangle\right],\end{aligned} \tag{2.57}$$

where $D_0$ is the tree-level propagator given in Eq. (2.52). As for the static ghost propagator, there are no terms of $\mathcal{O}(g)$, since the connected pieces of $\langle 0|\,A\,A\,|0\rangle^{(1)}$ are given by the expectation value of five gauge field operators, which vanishes.

The normalization factor [second term in the first line of Eq. (2.57)] cancels the disconnected piece of the first term in the second line. From Eq. (2.57) it is also clear that the contributions to $|0\rangle^{(2)}$ with more than two-gluon states do not contribute to $D(\mathbf{k})$ to the order considered.

The matrix elements in Eq. (2.57) can be straightforwardly evaluated using the results of Sec. 2.2. Moreover, the last term in the second line of Eq. (2.57) can be expressed in terms of the preceding one since

$${}^{(2)}\!\langle 0|\,A(1)\,A(2)\,|0\rangle = {}^{(2)}\!\langle 0|\,A^\dagger(-1)\,A^\dagger(-2)\,|0\rangle = \Big(\langle 0|\,A(-1)\,A(-2)\,|0\rangle^{(2)}\Big)^*.$$

For the gluon form factor $D_A(\mathbf{k})$ defined by

$$D^{ab}_{ij}(\mathbf{k}) =: \delta^{ab}\,t_{ij}(\mathbf{k})\,\frac{D_A(\mathbf{k})}{2|\mathbf{k}|} \tag{2.58}$$

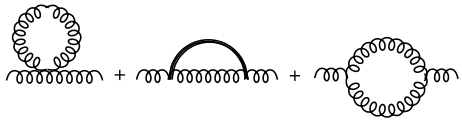

Figure 2.1: Diagrammatic representation of Eq. (2.59). The curly and the full lines represent, respectively, the static gluon and Coulomb propagators.

we get at one-loop level

$$\begin{aligned} D_A(\mathbf{k}) = 1 &- g^2\mu^{3-d}\,\frac{N_\mathrm{c}}{8(d-1)\mathbf{k}^2}\int \text{đ}q\,\frac{d^2-3d+3-(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})^2}{|\mathbf{q}|}+ \\ &+ g^2\mu^{3-d}\,\frac{N_\mathrm{c}}{4(d-1)}\int \text{đ}q\,\frac{t_{ij}(\mathbf{k})\,t_{ij}(\mathbf{q})}{(\mathbf{k}-\mathbf{q})^2}\left[\frac{1}{|\mathbf{q}|}-\frac{|\mathbf{q}|}{\mathbf{k}^2}\right]+ \\ &+ g^2\mu^{3-d}\,\frac{N_\mathrm{c}}{8(d-1)\mathbf{k}^2}\int \text{đ}q\,\text{đ}p\,\frac{T(\mathbf{p},\mathbf{k},\mathbf{q})(2\pi)^d\delta(\mathbf{p}+\mathbf{k}+\mathbf{q})}{(|\mathbf{p}|+|\mathbf{k}|+|\mathbf{q}|)\,|\mathbf{p}|\,|\mathbf{q}|} \\ &\qquad\qquad\times\left[1+\frac{|\mathbf{k}|}{|\mathbf{p}|+|\mathbf{k}|+|\mathbf{q}|}\right], \end{aligned} \tag{2.59}$$

with $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ and

$$T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \mathrm{tr}|T(1,2,3)|^2, \tag{2.60}$$

where $T(1,2,3)$ is defined in Eq. (2.17) and the trace is taken in Lorentz space, yielding

$$\begin{aligned} T(\mathbf{k},\mathbf{q},\mathbf{p}) = {}& 2(\mathbf{p}^2+\mathbf{q}^2+\mathbf{k}^2)\big[d-2-(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})(\hat{\mathbf{q}}\cdot\hat{\mathbf{k}})(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})\big]+ \\ &+2(d-1)\big[(\hat{\mathbf{p}}\cdot\mathbf{q})(\hat{\mathbf{p}}\cdot\mathbf{k})+(\hat{\mathbf{q}}\cdot\mathbf{p})(\hat{\mathbf{q}}\cdot\mathbf{k})+(\hat{\mathbf{k}}\cdot\mathbf{q})(\hat{\mathbf{k}}\cdot\mathbf{p})\big]+ \\ &-2(d-1)\big[\mathbf{p}\cdot\mathbf{q}+\mathbf{q}\cdot\mathbf{k}+\mathbf{p}\cdot\mathbf{k}\big]+ \\ &-(d-3)\big[(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2(\mathbf{p}^2+\mathbf{q}^2)+(\hat{\mathbf{q}}\cdot\hat{\mathbf{k}})^2(\mathbf{q}^2+\mathbf{k}^2)+(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^2(\mathbf{p}^2+\mathbf{k}^2)\big]. \end{aligned} \tag{2.61}$$

The diagrammatic representation of Eq. (2.59) is shown in Fig. 2.1.[1]

The first integral in Eq. (2.59) is a tadpole term, which vanishes identically in dimensional regularization. Moreover, in the last line of Eq. (2.59) we can carry out one of the momentum integrals due to the $\delta$ function, yielding

$$\begin{aligned} D_A(\mathbf{k}) &= 1 + g^2\,\mu^{3-d}\,\frac{N_\mathrm{c}}{4(d-1)\mathbf{k}^2}\int \text{đ}q\,\frac{d-2+(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})^2}{(\mathbf{k}-\mathbf{q})^2}\,\frac{\mathbf{k}^2-\mathbf{q}^2}{|\mathbf{q}|}+ \\ &\quad + g^2\,\mu^{3-d}\,\frac{N_\mathrm{c}}{2(d-1)\mathbf{k}^2}\int \text{đ}q\,\frac{\Sigma(\mathbf{k},\mathbf{q})}{|\mathbf{q}|\,|\mathbf{k}-\mathbf{q}|}\,\frac{2|\mathbf{k}|+|\mathbf{q}|+|\mathbf{k}-\mathbf{q}|}{(|\mathbf{k}|+|\mathbf{q}|+|\mathbf{k}-\mathbf{q}|)^2} \\ &=: 1 + I_c(\mathbf{k}) + I_g(\mathbf{k}), \end{aligned} \tag{2.62}$$

with

$$\begin{aligned} \Sigma(\mathbf{k},\mathbf{q}) &= t_{il}(\mathbf{k})\,t_{jm}(\mathbf{q})\,t_{kn}(\mathbf{k}-\mathbf{q})\big[\delta_{ij}k_k-\delta_{jk}q_i-\delta_{ik}k_j\big]\big[\delta_{lm}k_n-\delta_{mn}q_l-\delta_{nl}k_m\big] \\ &= \Big[1-(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})^2\Big]\left[(d-1)(\mathbf{k}^2+\mathbf{q}^2)+\frac{(d-2)\,\mathbf{k}^2\,\mathbf{q}^2+(\mathbf{k}\cdot\mathbf{q})^2}{(\mathbf{k}-\mathbf{q})^2}\right]. \end{aligned} \tag{2.63}$$

[1] Notice that these graphs are *not* standard Feynman diagrams.

The first integral in Eq. (2.62), i.e. the contribution $I_c(\mathbf{k})$ from the Coulomb kernel, can be evaluated in $d = 3 - 2\varepsilon$ dimensions by means of standard techniques, yielding

$$I_c(\mathbf{k}) = \frac{g^2\, N_\mathrm{c}}{(4\pi)^{2-\varepsilon}} \left[ \frac{4}{15} \left( \frac{1}{\varepsilon} - \ln \frac{\mathbf{k}^2}{\mu^2} - \gamma_\mathrm{E} \right) - \frac{8}{15}\, \ln 2 + \frac{188}{225} + \mathcal{O}(\varepsilon) \right]. \tag{2.64}$$

Unfortunately, the gluon loop $I_g(\mathbf{k})$ [second integral of Eq. (2.62)] is highly non-trivial. It would probably be possible to evaluate $I_g(\mathbf{k})$ using partial differential equations techniques similar to the ones used in Refs. [7,8]. Instead of using these techniques, we will show that the integrals in Eq. (2.62) are the same as the ones treated in Refs. [7,8], and we will use the result of those papers.

### 2.4.2 Static gluon propagator from the Lagrangian approach

In the Lagrangian-based functional integral approach to Yang–Mills theory in Coulomb gauge considered in Refs. [7,8], the full (energy-dependent) propagator has the form

$$\langle A_i^a(p) A_j^b(k) \rangle = (2\pi)^{d+1}\, \delta(p+k)\, \delta^{ab}\, t_{ij}(\mathbf{k})\, W(k_4, \mathbf{k}), \tag{2.65}$$

where $W(k_4, \mathbf{k})$ can be expressed (in Euclidean space) as

$$W(k_4, \mathbf{k}) = \frac{D_{AA}(k_4, \mathbf{k})}{k_4^2 + \mathbf{k}^2}\,. \tag{2.66}$$

Here we have introduced the dressing function $D_{AA}(k_4, \mathbf{k})$, which measures the deviation of the propagator from the tree-level form. We are interested here in the *static* or *equal-time* propagator, the quantity considered in the Hamiltonian approach, which is obtained from $W(k_4, \mathbf{k})$ by integrating out the temporal component of the 4-momentum

$$W(\mathbf{k}) = \int \frac{\mathrm{d}k_4}{2\pi}\, W(k_4, \mathbf{k}). \tag{2.67}$$

At tree-level (where $D_{AA} = 1$) this yields

$$W_0(\mathbf{k}) = \frac{1}{2|\mathbf{k}|}\,, \tag{2.68}$$

which is precisely the static tree-level gluon propagator of the Hamiltonian approach, see Eq. (2.52). For sake of comparison with the Hamiltonian approach [Eqs. (2.58) and (2.62)], we also express the full equal-time gluon propagator (2.67) by a dressing function $\bar{D}_{AA}(\mathbf{k})$

$$W(\mathbf{k}) =: \bar{D}_{AA}(\mathbf{k})\, W_0(\mathbf{k}) \overset{(2.68)}{=} \frac{\bar{D}_{AA}(\mathbf{k})}{2|\mathbf{k}|}\,. \tag{2.69}$$

The two dressing functions (form factors) in (2.66) and (2.69) are related by

$$\bar{D}_{AA}(\mathbf{k}) = 2|\mathbf{k}| \int \frac{\mathrm{d}k_4}{2\pi} \frac{D_{AA}(k_4, \mathbf{k})}{k_4^2 + \mathbf{k}^2}\,. \tag{2.70}$$

With the dressing function of the energy-dependent propagator $D_{AA}(k_4, \mathbf{k})$ given in Refs. [7,8], we can calculate the dressing function $\bar{D}_{AA}(\mathbf{k})$ of the equal-time propagator. We will

now show that the so obtained equal-time dressing function of the gluon propagator Eq. (2.70) coincides, at one-loop level, with the gluon form factor of the Hamiltonian approach, defined in Eq. (2.58).

In Refs. [7,8] the dressing function $D_{AA}(k_4, \mathbf{k})$ is evaluated at one-loop level, with the result[2]

$$\begin{aligned} D_{AA}(k_4,\mathbf{k}) = 1 &+ g^2\,\mu^{3-d}\,\frac{2N_\mathrm{c}}{(d-1)}\int\frac{\mathrm{d}^d q\;\mathrm{d}q_4}{(2\pi)^{d+1}}\frac{\Sigma(\mathbf{k},\mathbf{q})}{k^2\,q^2\,(k-q)^2} + \\ &+ g^2\,\mu^{3-d}\,\frac{N_\mathrm{c}}{(d-1)}\int\frac{\mathrm{d}^d q\;\mathrm{d}q_4}{(2\pi)^{d+1}}\frac{t_{ij}(\mathbf{k})t_{ij}(\mathbf{q})}{k^2\,q^2\,(\mathbf{k}-\mathbf{q})^2}\,(k_4^2-\mathbf{q}^2). \end{aligned} \tag{2.71}$$

Here $\Sigma$ is the kernel obtained by contracting the three-gluon vertex, defined in Eq. (2.63). Inserting Eq. (2.71) in Eq. (2.70) and also performing the loop integration over $q_4$ one obtains for the equal-time dressing function $\bar{D}_{AA}(\mathbf{k})$ [Eq. (2.70)] precisely the form factor $D_A(\mathbf{k})$ of the static gluon propagator of the Hamiltonian approach [Eq. (2.62)]. We have thus shown that the Lagrangian-based functional integral approach yields the same equal-time gluon propagator as the time-independent Hamiltonian approach, at least to the order considered. We have checked that this equivalence does also hold for the $\langle\Pi\Pi\rangle$ correlator, but it does *not* hold for the $\langle A\Pi\rangle$ correlator. The reason is that in the time-dependent Lagrangian approach the $\langle A\Pi\rangle$ correlator is odd under time reversal and thus the corresponding equal-time correlator vanishes, while in the Hamiltonian approach the static $\langle A\Pi\rangle$ correlator is constrained by the canonical commutation relation not to vanish.

In Refs. [7,8], the loop corrections to the gluon dressing function [Eq. (2.71)] were calculated in dimensional regularization. With $d = 3 - 2\varepsilon$ the result reads

$$\begin{aligned} D_{AA}(k_4,\mathbf{k}) = 1 + \frac{g^2\,N_\mathrm{c}}{(4\pi)^{2-\varepsilon}}\Bigg\{ &\left[\frac{1}{\varepsilon} - \gamma_\mathrm{E} - \ln\frac{\mathbf{k}^2}{\mu^2}\right] - \ln(1+z) - \frac{64}{9} + 3z + \\ &+ \frac{1}{4}\,f(z)\left[\frac{1}{z} - 1 - 11z - 3z^2\right] + g(z)\left[-\frac{1}{2z} + \frac{14}{3} - \frac{3}{2}\,z\right] + \mathcal{O}(\varepsilon)\Bigg\}, \end{aligned} \tag{2.72}$$

where $z = k_4^2/\mathbf{k}^2$ and the functions $f(z)$, $g(z)$ are defined by

$$f(z) = 4\,\ln 2\,\frac{\arctan\sqrt{z}}{\sqrt{z}} - \int_0^1 \mathrm{d}t\,\frac{\ln(1+z\,t)}{\sqrt{t}\,(1+z\,t)}\,, \tag{2.73a}$$

$$g(z) = 2\,\ln 2 - \ln(1+z). \tag{2.73b}$$

Using this result, after integrating Eq. (2.72) over $k_4$ with the appropriate tree-level factor, see Eq. (2.70), we find for the static gluon form factor (2.58)

$$D_A(\mathbf{k}) = 1 + g^2\,\frac{N_\mathrm{c}}{(4\pi)^{2-\varepsilon}}\left[\left(\frac{1}{\varepsilon} - \ln\frac{\mathbf{k}^2}{\mu^2}\right) + \cdots\right] \tag{2.74}$$

where the ellipsis contains the finite constant terms.

Finally, we remark that the ghost propagator in the Lagrangian-based functional integral approach [7,8] defined there as the correlator $\langle c\bar{c}\rangle$ for explicitly introduced ghost and antighost fields, is given by a function $W_c(k_4, \mathbf{k})$ in analogy with Eq. (2.65) for the gluon propagator, which is *independent* of the temporal component $k_4$, $W_c(k_4, \mathbf{k}) = W_c(\mathbf{k})$. It is easily seen, by integrating over the temporal component of the loop momentum, that $W_c(\mathbf{k})$ coincides with our result [Eq. (2.54)] for $G(\mathbf{k}) = D_c(\mathbf{k})/\mathbf{k}^2$.

[2] The results of [7] and [8], being evaluated in the, respectively, first and second order formalism, are at first sight not identical. However, it is not difficult to show that they indeed agree.

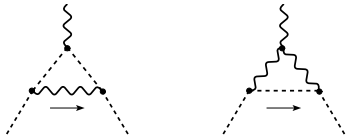

Figure 2.2: Diagrammatic representation of the one-loop correction to the ghost-gluon vertex. The left and right picture correspond to, respectively, the first and second integral in Eq. (2.77). The arrow shows the flow of the loop momentum $\boldsymbol{\ell}$.

## 2.5 The ghost-gluon vertex and the $\beta$ function

In the Hamiltonian approach the ghost-gluon vertex is given by [18]

$$\langle\Omega|\, A_i^a(\mathbf{x})\, G_A^{bc}(\mathbf{y}_1,\mathbf{y}_2)\,|\Omega\rangle =: \int \mathrm{d}^d z_1\, \mathrm{d}^d z_2\, \mathrm{d}^d z_3\, D_{ij}(\mathbf{x},\mathbf{z}_1)\, G(\mathbf{y}_1,\mathbf{z}_2)\, G(\mathbf{y}_2,\mathbf{z}_3)\, \Gamma_i^{abc}(\mathbf{z}_1;\mathbf{z}_2,\mathbf{z}_3), \quad (2.75)$$

where $D$ is the full gluon propagator of Sec. 2.4.1 and $G$ the static ghost propagator as defined in Sec. 2.3. Equation (2.75) tells us to calculate the vacuum expectation value $\langle AG_A\rangle$ and then "cut off" the external legs, in order to extract the irreducible component. In Eq. (2.75) we already used the fact that ghost and gluon propagators are colour diagonal to every order in perturbation theory.

The lowest-order contribution to Eq. (2.75) is the bare vertex $\Gamma^{(0)}$, given in momentum space by

$$\Gamma_i^{(0)abc}(\mathbf{k};\mathbf{p},\mathbf{q}) = -\mathrm{i}\, g\, f^{abc} p_i\,, \quad (2.76)$$

(all momenta defined as incoming). With arguments similar to the ones we used in the evaluation of the ghost and gluon propagators, it is not difficult to show that the terms contributing to the next non-vanishing order are

$$\langle 0|\, AG_2[A]\,|0\rangle^{(1)} + {}^{(1)}\langle 0|\, AG_2[A]\,|0\rangle + \langle 0|\, AG_3[A]\,|0\rangle_{1\mathrm{PI}}\,,$$

where the subscript '1PI' means that only the irreducible terms have to be considered. The evaluation of these matrix elements is straightforward and yields

$$\Gamma_i^{(2)abc}(\mathbf{k};\mathbf{p},\mathbf{q}) = -\mathrm{i}\, g^3\, f^{abc} \frac{N_\mathrm{c}}{4} \left\{ \int \text{đ}\ell\, \frac{(\ell-p)_i\, p_j\, q_m\, t_{jm}(\boldsymbol{\ell})}{|\boldsymbol{\ell}|(\mathbf{p}-\boldsymbol{\ell})^2(\mathbf{q}+\boldsymbol{\ell})^2} + \right. \\ \left. + \int \text{đ}\ell\, \frac{p_j\, q_m}{\boldsymbol{\ell}^2 |\mathbf{p}-\boldsymbol{\ell}|\,|\mathbf{q}+\boldsymbol{\ell}|} \frac{T_{ijm}(\mathbf{k},\mathbf{p}-\boldsymbol{\ell},\mathbf{q}+\boldsymbol{\ell})}{|\mathbf{k}|+|\mathbf{p}-\boldsymbol{\ell}|+|\mathbf{q}+\boldsymbol{\ell}|} \right\}, \quad (2.77)$$

where $T$ is the Lorentz structure of the three-gluon vertex Eq. (2.17). The diagrammatic representation of Eq. (2.77) is given in Fig. 2.2. The integrals in Eq. (2.77) are UV finite in $d = 3$ spatial dimensions, hence they are independent of the scale when evaluated at a symmetry point (after factorizing the momentum that carries the Lorentz index $i$). Then $g^2\, D_A\, D_c^2$ is a renormalization group invariant, at least to the present order. From Eqs. (2.55) and (2.74) we obtain in Coulomb gauge

$$D_A(\mathbf{k}) D_c^2(\mathbf{k}) = \left[1 + g^2\, \frac{N_\mathrm{c}}{(4\pi)^{2-\varepsilon}}\, \frac{1}{\varepsilon} + \ldots\right] \left[1 + g^2\, \frac{N_\mathrm{c}}{(4\pi)^{2-\varepsilon}}\, \frac{4}{3}\, \frac{1}{\varepsilon} + \ldots\right]^2 = \\ = 1 + g^2\, \frac{N_\mathrm{c}}{(4\pi)^{2-\varepsilon}}\, \frac{11}{3}\, \frac{1}{\varepsilon} + \ldots$$

Consequently, $g^2$ must have a $1/\varepsilon$ pole with coefficient $(-11/3)N_\mathrm{c}$, and since, at the one-loop level, the coefficient of the $1/\varepsilon$ pole of $g^2$ is the (gauge invariant) first coefficient $\beta_0$ of the $\beta$ function,

$$\beta(g) = \frac{\partial g}{\partial \ln \mu} = \frac{1}{(4\pi)^2}\,\beta_0\, g^3 + \mathcal{O}(g^5), \tag{2.78}$$

we find

$$\beta_0 = -\frac{11}{3}\,N_\mathrm{c}\,, \tag{2.79}$$

which is the correct value.

## 2.6 The potential for static sources

Until now, we have considered pure Yang–Mills theory without external colour charges. If static external sources are included, the Hamilton operator becomes

$$\begin{aligned}\widetilde{H} = \widetilde{H}_\mathrm{YM} &+ \frac{g^2}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\rho_m^a(\mathbf{x}) F_A^{ab}(\mathbf{x},\mathbf{y})\rho_m^b(\mathbf{y}) + \\ &+ \frac{g^2}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\left[\rho_m^a(\mathbf{x}) F_A^{ab}(\mathbf{x},\mathbf{y})\hat{A}_i^{bc}(\mathbf{y})\widetilde{\Pi}_i^c(\mathbf{y}) + \hat{A}_i^{ac}(\mathbf{x})\widetilde{\Pi}_i^{c\dagger}(\mathbf{x}) F_A^{ab}(\mathbf{x},\mathbf{y})\rho_m^b(\mathbf{y})\right],\end{aligned} \tag{2.80}$$

where $\widetilde{H}_\mathrm{YM}$ is the Hamiltonian [Eq. (2.9)] of the Yang–Mills sector considered in the previous sections, and $\rho_m^a(\mathbf{x})$ is the external charge density. Since

$$\widetilde{\Pi}_i^a(\mathbf{x}) = \Pi_i^a(\mathbf{x}) + \frac{\mathrm{i}}{2}\frac{\delta \ln \mathcal{J}_A}{\delta A_i^a(\mathbf{x})}\,, \qquad \widetilde{\Pi}_i^{a\dagger}(\mathbf{x}) = \Pi_i^a(\mathbf{x}) - \frac{\mathrm{i}}{2}\frac{\delta \ln \mathcal{J}_A}{\delta A_i^a(\mathbf{x})}\,,$$

the $\delta \ln \mathcal{J}_A/\delta A$ contributions cancel in the terms linear in the external charge density and we can omit the tildes there, obtaining

$$\begin{aligned}\widetilde{H} = \widetilde{H}_\mathrm{YM} &+ \frac{g^2}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\rho_m^a(\mathbf{x}) F_A^{ab}(\mathbf{x},\mathbf{y})\rho_m^b(\mathbf{y}) + \\ &+ \frac{g^2}{2}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\left\{\rho_m^a(\mathbf{x}) F_A^{ab}(\mathbf{x},\mathbf{y}), \hat{A}_i^{bc}(\mathbf{y})\,\Pi_i^c(\mathbf{y})\right\}.\end{aligned} \tag{2.81}$$

($\{,\}$ denotes the anticommutator.) We are interested here in a perturbative calculation of the static potential, i.e. the potential between static charges. For this purpose, we will follow the approach of Refs. [19–21], treating the external sources as perturbations to the pure Yang–Mills sector. Suppose we know the exact spectrum of $\widetilde{H}_\mathrm{YM}$, i.e. its eigenstates $|\Phi_N\rangle$ and the corresponding eigenvalues $E_{\Phi_N}$. Second-order perturbation theory in the external sources then yields for the potential

$$V^{ab}(\mathbf{x},\mathbf{y}) = \langle\Phi_0|\,F_A^{ab}(\mathbf{x},\mathbf{y})\,|\Phi_0\rangle - \frac{g^2}{2}\sum_{N\neq 0}\frac{\langle\Phi_0|\,K^a(\mathbf{x})\,|\Phi_N\rangle\langle\Phi_N|\,K^b(\mathbf{y})\,|\Phi_0\rangle}{E_{\Phi_N} - E_{\Phi_0}}\,, \tag{2.82}$$

where we have introduced the quantity

$$K^a(\mathbf{x}) := \int \mathrm{d}^d z\,\left\{F_A^{ab}(\mathbf{x},\mathbf{z}), \hat{A}_i^{bc}(\mathbf{z})\Pi_i^c(\mathbf{z})\right\}. \tag{2.83}$$

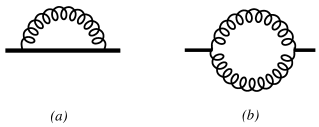


Figure 2.3: Diagrams contributing to the static potential in second-order perturbation theory. *(a)* and *(b)* represent, repectively, the second term of Eq. (2.86), and Eq. (2.84).

Since we are interested in the potential to order $\mathcal{O}(g^2)$, we can replace all quantities in the second term in Eq. (2.82) by their unperturbed expressions. This yields

$$-\frac{g^2}{2}\sum_{N\neq 0}\frac{\langle\Phi_0|\,K^a(\mathbf{x})\,|\Phi_N\rangle\langle\Phi_N|\,K^b(\mathbf{y})\,|\Phi_0\rangle}{E_{\Phi_N}-E_{\Phi_0}} = -g^2\sum_{1,2}\frac{|\langle 0|\,F_0\hat{A}\Pi\,|g_1,g_2\rangle|^2}{|\mathbf{k}_1|+|\mathbf{k}_2|}+\mathcal{O}(g^4), \quad (2.84)$$

where the unperturbed part of $F_A^{ab}(\mathbf{x},\mathbf{y})$ [Eq. (2.5)] in $d=3$ spatial dimensions

$$F_0^{ab}(\mathbf{x},\mathbf{y}) = \delta^{ab}\,\left[(-\partial^2)^{-1}\right]_{\mathbf{x},\mathbf{y}} = \frac{\delta^{ab}}{4\pi|\mathbf{x}-\mathbf{y}|} \quad (2.85)$$

is the familiar Coulomb interaction.

In the first term of Eq. (2.82) (the so-called colour Coulomb potential) we use the perturbative expansion of the vacuum wave functional $|\Omega\rangle$ [Eq. (2.44)], and also expand the Coulomb kernel $F_A^{ab}(\mathbf{x},\mathbf{y})$ [Eq. (2.5)] in powers of $g$. Thereby many terms can be shown to vanish or cancel using similar arguments as in the evaluation of the static ghost propagator. In the end one obtains for the Coulomb potential

$$\langle\Omega|\,F_A^{ab}(\mathbf{x},\mathbf{y})\,|\Omega\rangle = F_0^{ab}(\mathbf{x},\mathbf{y}) + g^2\,\langle 0|\,F_2^{ab}[A](\mathbf{x},\mathbf{y})\,|0\rangle + \mathcal{O}(g^4), \quad (2.86)$$

where $F_2^{ab}[A](\mathbf{x},\mathbf{y}) = 3G_2^{ab}[A](\mathbf{x},\mathbf{y})$ is the $\mathcal{O}(g^2)$ part of the Coulomb kernel $F_A$ [Eq. (2.5)].

For the potential dressing function $v(\mathbf{k})$ defined by

$$V^{ab}(\mathbf{x},\mathbf{y}) = \int \text{đ}k\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{\delta^{ab}\,v(\mathbf{k})}{\mathbf{k}^2} \quad (2.87)$$

we obtain the following expression

$$\begin{aligned} v(\mathbf{k}) = 1 &+ g^2\mu^{3-d}\frac{3N_\mathrm{c}}{2\mathbf{k}^2}\int \text{đ}q\,\frac{k_i k_j t_{ij}(\mathbf{q})}{(\mathbf{k}-\mathbf{q})^2|\mathbf{q}|} + \\ &- g^2\mu^{3-d}\frac{N_\mathrm{c}}{2\mathbf{k}^2}\int \text{đ}q\,\frac{t_{ij}(\mathbf{q})\,t_{ij}(\mathbf{k}-\mathbf{q})}{|\mathbf{q}|}\,\frac{|\mathbf{k}-\mathbf{q}|-|\mathbf{q}|}{|\mathbf{k}-\mathbf{q}|+|\mathbf{q}|}\,. \end{aligned} \quad (2.88)$$

The diagrammatic representation of this equation is given in Fig. 2.3. The first integral of Eq. (2.88) comes from the expectation value of $F_2$ [second term in Eq. (2.86)], while the second integral arises from the second-order term [Eq. (2.84)].

While the first integral in Eq. (2.88) is standard, the second one is of the same type we encountered in the calculation of the static gluon propagator. Again, we will use the results of the Lagrangian-based functional integral approach. As argued by Zwanziger [22],

the potential between static colour charges should be related to the correlation function of the $A_0$ field. For the corresponding dressing function $D_\sigma$ defined by

$$\langle A_0^a(p) A_0^b(k)\rangle = (2\pi)^{d+1}\,\delta(p+k)\,\delta^{ab}\,\frac{D_\sigma(k_4,\mathbf{k})}{\mathbf{k}^2}\,, \tag{2.89}$$

the result at one-loop order reads [7]

$$\begin{aligned} D_\sigma(k_4,\mathbf{k}) = 1 &+ g^2\mu^{3-d}\,\frac{3N_\mathrm{c}}{2\mathbf{k}^2}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{k_i k_j t_{ij}(\mathbf{q})}{q^2(\mathbf{k}-\mathbf{q})^2} + \\ &+ g^2\mu^{3-d}\,\frac{N_\mathrm{c}}{2\mathbf{k}^2}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{(\mathbf{k}^2-2\mathbf{k}\cdot\mathbf{q})\,t_{ij}(\mathbf{q})\,t_{ij}(\mathbf{k}-\mathbf{q})}{|\mathbf{k}-\mathbf{q}|\left[k_4^2+(|\mathbf{k}-\mathbf{q}|+|\mathbf{q}|)^2\right]}\,, \end{aligned} \tag{2.90}$$

where we have already performed the integration over the temporal component $q_4$ of the loop momentum. Noticing that

$$\mathbf{k}^2 - 2\mathbf{k}\cdot\mathbf{q} = (\mathbf{k}-\mathbf{q})^2 - \mathbf{q}^2 = \big(|\mathbf{k}-\mathbf{q}|+|\mathbf{q}|\big)\big(|\mathbf{k}-\mathbf{q}|-|\mathbf{q}|\big), \tag{2.91}$$

we see that at one-loop level

$$v(\mathbf{k}) = D_\sigma(k_4=0,\mathbf{k}). \tag{2.92}$$

Taking then the result for $D_\sigma$ obtained in [7]

$$\begin{aligned} D_\sigma(k_4,\mathbf{k}) = 1 + \frac{g^2 N_\mathrm{c}}{(4\pi)^{2-\varepsilon}}\Bigg\{ &\frac{11}{3}\left[\frac{1}{\varepsilon} - \gamma_\mathrm{E} - \ln\frac{\mathbf{k}^2}{\mu^2} - \ln(1+z)\right] + \frac{31}{9} - 6z + \\ &+ (3z-1)g(z) + \frac{1}{2}(1+z)(1+3z)f(z) + \mathcal{O}(\varepsilon)\Bigg\}\,, \end{aligned} \tag{2.93}$$

where $z = k_4^2/\mathbf{k}^2$ and the functions $f(z)$, $g(z)$ are defined in Eqs. (2.73), we get for the dressing function of the static potential

$$v(\mathbf{k}) = 1 + g^2\,\frac{N_\mathrm{c}}{(4\pi)^{2-\varepsilon}}\left\{\frac{11}{3}\left[\frac{1}{\varepsilon} - \gamma_\mathrm{E} - \ln\frac{\mathbf{k}^2}{\mu^2}\right] + \frac{31}{9} + \mathcal{O}(\varepsilon)\right\}. \tag{2.94}$$

A few remarks are in order here. The equivalence between the static potential evaluated in the Hamiltonian approach and the propagator of the $A_0$ field in the Lagrangian-based formalism is far from trivial, since in the Hamiltonian approach the Weyl gauge $A_0^a = 0$ is also imposed. Furthermore, note that the Hamiltonian static potential is not obtained as the equal-time component, but rather as the integral over the relative time of the $\langle A_0 A_0\rangle$ correlator. Such an integral captures the instantaneous interaction through the first correction term in Eq. (2.90) as much as the retarded interaction via vacuum polarization given by the second correction term there.

Given the fact that the physical potential is $g^2 V^{ab}$, with $V^{ab}$ given by Eq. (2.87), $v(\mathbf{k})$ in Eq. (2.88) represents the form factor of the running coupling. From Eq. (2.94) we then find $\beta_0 = -11N_\mathrm{c}/3$, which is again the correct coefficient. Since this agrees with $D_A(\mathbf{p})D_c^2(\mathbf{p})$, we find the same running coupling from the ghost-gluon vertex and the Coulomb potential, at least to the order considered.

## 2.7 Relation with the variational approach

In the variational approach considered in Refs. [2,5], a generalization of Eq. (2.20) was taken for the vacuum functional,

$$\langle A|\omega\rangle = \mathcal{N} \exp\left\{-\frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\, A_i^a(\mathbf{k})\, t_{ij}(\mathbf{k})\, \omega(\mathbf{k})\, A_j^a(-\mathbf{k})\right\}, \tag{2.95}$$

and the kernel $\omega(\mathbf{k})$ was determined by minimization of the vacuum energy density. In this non-perturbative case a complete basis can also be defined by Eqs. (2.30) and (2.33) with $|\mathbf{k}|$ replaced by $\omega(\mathbf{k})$. Considering the three-gluon vertex as a perturbation on top of the non-perturbative vacuum Eq. (2.95), the energy functional gets an additional contribution

$$\Delta E[\omega] = -\frac{g^2}{3!}\sum_{1,2,3} \frac{|\langle\omega|\,\widetilde{H}_1\,|g_1,g_2,g_3\rangle|^2}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)+\omega(\mathbf{k}_3)}\,. \tag{2.96}$$

Functional differentiation of this expression with respect to $\omega(\mathbf{k})$ yields an additional term to the gap equation

$$\begin{aligned}\omega^2(\mathbf{k}) = \mathbf{k}^2 + \chi^2(\mathbf{k}) + I_\omega^0 + I_\omega(\mathbf{k}) + \\ - g^2\,\frac{N_\mathrm{c}}{8}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{T(\mathbf{k},\mathbf{q},\mathbf{p})\,(2\pi)^3\delta(\mathbf{p}+\mathbf{q}+\mathbf{k})}{[\omega(\mathbf{k})+\omega(\mathbf{q})+\omega(\mathbf{p})]\,\omega(\mathbf{p})\,\omega(\mathbf{q})} \\ \times\,\frac{2\omega(\mathbf{k})+\omega(\mathbf{q})+\omega(\mathbf{p})}{\omega(\mathbf{k})+\omega(\mathbf{q})+\omega(\mathbf{p})}\,,\end{aligned} \tag{2.97}$$

where $T(\mathbf{k},\mathbf{q},\mathbf{p})$ is given in Eq. (2.60) and the integral terms $\chi(\mathbf{k})$, $I_\omega^0$, and $I_\omega(\mathbf{k})$ are defined in Ref. [6]. For $\mathbf{k}\to\infty$ this additional term reduces to the integral $I_g(\mathbf{k})$, Eq. (2.62), found in perturbation theory, showing that Eq. (2.97) does indeed provide the right contribution to the variational form of the gap equation to produce the correct UV asymptotic behaviour, at the same time leaving the ghost dominated infrared sector untouched. The numerical solution of this modified gap equation is in progress.

## 2.8 Summary and Conclusions

In this paper we have studied the Hamiltonian approach to Yang–Mills theory in Coulomb gauge in Rayleigh–Schrödinger perturbation theory. The static gluon and ghost propagators as well as the potential between static colour sources have been calculated to one-loop order using dimensional regularization. The one-loop $\beta$ function was calculated from the ghost-gluon vertex as well as from the static potential. In both cases the result known from covariant perturbation theory was reproduced. The unperturbed basis constructed from the eigenstates of the unperturbed Hamiltonian, which up to the colour index of the gauge field coincides with the Hamiltonian for QED (without fermions), was generalized to multiple quasi-gluon states on top of the non-perturbative vacuum wave functional used in the variational approach [2,5]. Treating the three-gluon vertex, which is not captured by the variational wave functional used so far, as a perturbation on top of the non-perturbative vacuum, a modified gap equation was derived, which yields the correct (perturbative) ultraviolet asymptotics for the static propagators known from perturbation theory while,

at the same time, leaving the (non-perturbative) infrared behaviour of the propagator unchanged. The multiple quasi-gluon basis constructed on top of the non-perturbative vacuum wave functional will also serve as a basis for calculating the partition function of Yang–Mills theory in the Hamiltonian approach in Coulomb gauge and investigating the deconfinement phase transition at finite temperatures.

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Mus non uno fidit antro.

*(Titus Maccius Plautus)*

# Article 3

# Equal-time two-point correlation functions in Coulomb gauge Yang–Mills theory

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**Abstract** We apply a functional perturbative approach to the calculation of the equal-time two-point correlation functions and the potential between static colour charges to one-loop order in Coulomb gauge Yang–Mills theory. The functional approach proceeds through a solution of the Schrödinger equation for the vacuum wave functional to order $g^2$ and derives the equal-time correlation functions from a functional integral representation via new diagrammatic rules. We show that the results coincide with those obtained from the usual Lagrangian functional integral approach, extract the beta function, and determine the anomalous dimensions of the equal-time gluon and ghost two-point functions and the static potential under the assumption of multiplicative renormalizability to all orders.

## 3.1 Introduction

Finding an accurate (semi-)analytical description of the infrared sector of QCD is still one of the most important challenges of present-day quantum field theory. In this work we concentrate on Yang–Mills theory, QCD without dynamical quarks, since it is in this sector where the peculiar properties of QCD, in particular the confining interaction between quarks, arise. Recently, much of the activity in this area has focused on the formulation and (approximate) solution of Yang–Mills theory in the Coulomb gauge [1–11], the primary reason being that the Coulomb gauge Hamiltonian explicitly contains the colour-Coulomb potential which furnishes the dominant non-perturbative contribution to the static or heavy quark potential.

Semi-analytical functional approaches to the calculation of gluon and ghost propagators in the infrared, mostly using Dyson–Schwinger equations, have been successful in Landau gauge Yang–Mills theory [12–14]. In the so-called ghost dominance approximation, even

very simple analytical solutions exist in the far infrared [7,15–17]. Although the consistency of these solutions is still under discussion, it is natural to inquire whether a similar approach could be useful in the Coulomb gauge. The breaking of Lorentz covariance through the Coulomb gauge condition makes the usual Lagrangian functional integral approach quite cumbersome in this gauge, see e.g. Ref. [18]. For this reason, semi-analytical approaches in Coulomb gauge have mostly used a Hamiltonian formulation. A set of equations similar to Dyson–Schwinger equations is obtained from a variational principle using a Gaussian type of ansatzes for the vacuum wave functional in the Schrödinger representation [1,3–6]. In the ghost dominance approximation, furthermore, simple analytical solutions are available for the far infrared [2,7].

Nevertheless, the status of the semi-analytical and analytical solutions in the Coulomb gauge is not yet entirely clear, for two reasons: first, two different solutions with an infrared scaling behavior (differing in the infrared exponents) have been found in both the analytical and the semi-analytical approaches [2–7], and there is as yet no theoretical guidance to what the physical solution should be; second, the inclusion of the Coulomb form factor (the form factor for the colour-Coulomb potential, which measures the deviation of the Coulomb potential from a factorization in terms of ghost propagators) in the set of equations of Dyson–Schwinger type results problematic. In Refs. [3–6], the equation for the Coulomb form factor has been considered subleading compared to the equations for the gluon and ghost propagators and therefore treated in the tree-level approximation, while in Ref. [19] all equations have been considered to be of the same order and therefore treated on an equal footing, with the result that solutions with infrared scaling behavior cease to exist. It should be emphasized that only solutions with scaling behavior can give rise to a linearly rising Coulomb potential, and that the latest lattice calculations also show a scaling behavior for the equal-time correlation functions in the deep infrared [8,9]. It is not clear at present how to improve the approximation used in the variational approach in order to arrive at a unique and consistent solution.

An interesting relation between Landau and Coulomb gauge Yang–Mills theory has been pointed out in the ghost dominance approximation in Refs. [2,7]: the equal-time correlation functions of the Hamiltonian approach in Coulomb gauge are the formal counterparts in three dimensions of the covariant correlation functions in Landau gauge in four dimensions. Building on this analogy, a possible strategy would be to replace the variational principle by a calculation of equal-time correlation functions in the Coulomb gauge and intend to formulate Dyson–Schwinger equations for the latter. In the present work, we take a first step in this direction: we set up a functional integral representation of the equal-time correlation functions (without taking a detour to the space-time correlation functions) that is the precise three-dimensional analogue of the usual functional integral representation of the covariant correlation functions in the Lagrangian approach to Landau gauge Yang–Mills theory. We also develop a diagrammatic representation and a set of Feynman rules for the equal-time correlation functions. We use this formulation to calculate the equal-time gluon and ghost two-point correlation functions and the potential for static colour charges in Coulomb gauge perturbatively to one-loop order. We extract the one-loop beta function and determine the asymptotic ultraviolet behavior of the equal-time two-point functions and the static potential. We also show that our results coincide with those obtained in a Lagrangian functional integral approach [20,21] and use the latter for the renormalization of the equal-time correlation functions and the static potential.

The organization of the paper is as follows: in the next section, we determine the

vacuum wave functional perturbatively to order $g^2$ from the solution of the Schrödinger equation. With the vacuum functional determined to the corresponding order, we turn to the calculation of the equal-time gluon and ghost two-point correlation functions in Section 3.3. We also calculate the one-loop corrections to the static or heavy quark potential (and thus to the Coulomb form factor) in the same section. Although for the latter calculation we need to go beyond the terms that we have calculated for the vacuum functional in Section 3.2, the relevant additional contributions are quite simply determined. In Section 3.4, we provide another representation of the equal-time two-point functions by choosing equal times (zero) in the space-time correlation functions determined before in the Lagrangian functional integral representation [20,21] of the theory. The static potential can also be obtained from a two-point function that arises in the Lagrangian approach. We use the alternative representations of the two-point functions and the static potential to perform the renormalization of our results. We show the non-renormalization of the ghost-gluon vertex in the same section and use it to determine the beta function and the asymptotic ultraviolet behavior of the two-point functions. We also show that the same beta function is found from considering the static potential. Finally, in Section 3.5, we summarize our findings and comment on several possible applications. In the Appendix, we give some details on an important difference that arises between the Lagrangian and the Hamiltonian approach when it comes to the implementation of the Coulomb gauge.

## 3.2 Perturbative vacuum functional

It is very simple to write down a functional integral representation of the equal-time correlation functions, given that they are nothing but the vacuum expectation values of products of the field operators. In the Schrödinger representation of Yang–Mills theory in Coulomb gauge, the equal-time $n$-point correlation functions in (3-)momentum space have the following representation:

$$\begin{aligned}\langle A_i^a(\mathbf{p}_1, t=0) A_j^b(\mathbf{p}_2, t=0) \cdots A_r^f(\mathbf{p}_n, t=0)\rangle \\ = \int D[\mathbf{A}]\, \delta(\boldsymbol{\nabla}\cdot\mathbf{A})\, \mathsf{FP}(\mathbf{A})\, A_i^a(\mathbf{p}_1) A_j^b(\mathbf{p}_2)\cdots A_r^f(\mathbf{p}_n)\, |\psi(\mathbf{A})|^2. \quad (3.1)\end{aligned}$$

Here, $\psi(\mathbf{A})$ is the true vacuum wave functional of the theory. The (absolute) square $|\psi(\mathbf{A})|^2$ then plays the role of the exponential of the negative Euclidean classical action in the corresponding representation of the covariant correlation functions (in Euclidean space). $\mathsf{FP}(\mathbf{A}) \equiv \mathrm{Det}[-\boldsymbol{\nabla}\cdot\mathbf{D}(\mathbf{A})]$, with the covariant derivative in the adjoint representation defined as

$$\mathbf{D}^{ab}(\mathbf{A}) = \delta^{ab}\,\boldsymbol{\nabla} + g f^{abc}\mathbf{A}^c, \quad (3.2)$$

is the Faddeev–Popov determinant (in 3 dimensions) which forms a part of the integration measure for the scalar product of states in the Schrödinger representation (see Ref. [22]). Note that the fields $A_i^a(\mathbf{p})$ on the left-hand side of Eq. (3.1) are spatially transverse, $\mathbf{p}\cdot\mathbf{A}^a(\mathbf{p}) = 0$. We will assume the transversality of the fields $\mathbf{A}^a$ in all of the following formulae, which we could make manifest by introducing a transverse basis in momentum space. However, there is usually no need to do so explicitly.

In order to write down the functional integral for the equal-time correlation functions explicitly, the vacuum wave functional needs to be specified. The analogy with the covariant theory suggests to use an exponential ansatz for this wave functional, in the spirit

of the $e^S$ expansion in many-body physics [23]. We consider a full Volterra expansion of the exponent:

$$\psi(\mathbf{A}) = \exp\Bigg( -\sum_{k=2}^{\infty} \frac{1}{k!} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 p_k}{(2\pi)^3} \sum_{i_1, i_2, \ldots, i_k} \sum_{a_1, a_2, \ldots, a_k} f^{a_1 a_2 \ldots a_k}_{k; i_1 i_2 \ldots i_k}(-\mathbf{p}_1, \ldots, -\mathbf{p}_k) \times A^{a_1}_{i_1}(\mathbf{p}_1) \cdots A^{a_k}_{i_k}(\mathbf{p}_k) (2\pi)^3 \delta(\mathbf{p}_1 + \ldots + \mathbf{p}_k) \Bigg). \quad (3.3)$$

Any normalization factor can be conveniently absorbed in the functional integration measure in Eq. (3.1). Terms linear in $\mathbf{A}$ in the exponent ($k = 1$) are excluded by the symmetry of the wave functional under global gauge transformations (in the absence of external colour charges). Regarding notation, given that our Hamiltonian formalism is not manifestly covariant, we will denote the contravariant spatial components of 4-vectors by (latin) *subindices*.

We insert this ansatz for the vacuum wave functional into the Schrödinger equation

$$H\psi(\mathbf{A}) = E_0 \psi(\mathbf{A}), \quad (3.4)$$

where $H$ is the Christ–Lee Hamiltonian for Coulomb gauge Yang–Mills theory [22],

$$H = \frac{1}{2} \int \mathrm{d}^3 x \left( -\frac{1}{\mathsf{FP}(\mathbf{A})} \frac{\delta}{\delta A^a_i(\mathbf{x})} \mathsf{FP}(\mathbf{A}) \frac{\delta}{\delta A^a_i(\mathbf{x})} + B^a_i(\mathbf{x}) B^a_i(\mathbf{x}) \right) + \frac{g^2}{2} \int \mathrm{d}^3 x \, \mathrm{d}^3 y \, \frac{1}{\mathsf{FP}(\mathbf{A})} \, \rho^a(\mathbf{x}) \, \mathsf{FP}(\mathbf{A}) \, \langle \mathbf{x}, a | \, (-\boldsymbol{\nabla} \cdot \mathbf{D})^{-1} (-\nabla^2) (-\boldsymbol{\nabla} \cdot \mathbf{D})^{-1} \, | \mathbf{y}, b \rangle \, \rho^b(\mathbf{y}). \quad (3.5)$$

Here,

$$B^a_i = -\frac{1}{2} \varepsilon_{ijk} F^a_{jk} = \left( \boldsymbol{\nabla} \times \mathbf{A}^a - \frac{g}{2} f^{abc} \mathbf{A}^b \times \mathbf{A}^c \right)_i \quad (3.6)$$

is the chromo-magnetic field, and

$$\rho^a(\mathbf{x}) = \rho^a_q(\mathbf{x}) + f^{abc} A^b_j(\mathbf{x}) \, \frac{1}{\mathrm{i}} \frac{\delta}{\delta A^c_j(\mathbf{x})} \quad (3.7)$$

the colour charge density, including external static charges $\rho_q$ for later use. Note that we have extracted a factor $g$ from the colour charges in order to simplify the counting of orders of $g$ in the rest of the paper. The notation $\langle \mathbf{x}, a | \, C \, | \mathbf{y}, b \rangle$ refers to the kernel of the operator $C$ in an integral representation.

In most of the following perturbative calculation, we will need the Hamiltonian only up to order $g^2$, where

$$H = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left( -(2\pi)^3 \frac{\delta}{\delta A^a_i(\mathbf{p})} (2\pi)^3 \frac{\delta}{\delta A^a_i(-\mathbf{p})} + A^a_i(-\mathbf{p}) \, \mathbf{p}^2 A^a_i(\mathbf{p}) \right) \quad (3.8)$$

$$+\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,A_i^a(-\mathbf{p})\left(\frac{N_\mathrm{c}g^2}{2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2}{(\mathbf{p}-\mathbf{q})^2}\right)(2\pi)^3\frac{\delta}{\delta A_i^a(-\mathbf{p})} \tag{3.9}$$

$$+\frac{g}{3!}\int\frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{(2\pi)^3}\,\mathrm{i}f^{abc}\left[\delta_{jk}(p_{1,l}-p_{2,l})+\delta_{kl}(p_{2,j}-p_{3,j})+\delta_{lj}(p_{3,k}-p_{1,k})\right]$$
$$\times A_j^a(\mathbf{p}_1)A_k^b(\mathbf{p}_2)A_l^c(\mathbf{p}_3)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \tag{3.10}$$

$$+\frac{g^2}{4!}\int\frac{\mathrm{d}^3p_1}{(2\pi)^3}\cdots\frac{\mathrm{d}^3p_4}{(2\pi)^3}\Big[f^{abe}f^{cde}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+f^{ace}f^{bde}(\delta_{ij}\delta_{kl}-\delta_{il}\delta_{jk})$$
$$+f^{ade}f^{bce}(\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl})\Big]\,A_i^a(\mathbf{p}_1)A_j^b(\mathbf{p}_2)A_k^c(\mathbf{p}_3)A_l^d(\mathbf{p}_4)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4) \tag{3.11}$$

$$+\frac{g^2}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\rho^a(-\mathbf{p})\frac{1}{\mathbf{p}^2}\,\rho^a(\mathbf{p})+\mathcal{O}(g^3). \tag{3.12}$$

The term (3.9) stems from the application of the functional derivative to the Faddeev–Popov determinant. In this term, $N_\mathrm{c}$ stands for the number of colours, $f^{acd}f^{bcd}=N_\mathrm{c}\delta^{ab}$, and $\hat{\mathbf{p}}\equiv\mathbf{p}/|\mathbf{p}|$ denotes a unit vector. In the *absence* of external charges, we get for the term (3.12)

$$\frac{g^2}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\rho^a(-\mathbf{p})\frac{1}{\mathbf{p}^2}\,\rho^a(\mathbf{p})$$
$$=\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,A_i^a(-\mathbf{p})\left(N_\mathrm{c}g^2\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{t_{ij}(\mathbf{q})}{(\mathbf{p}-\mathbf{q})^2}\right)(2\pi)^3\frac{\delta}{\delta A_j^a(-\mathbf{p})} \tag{3.13}$$

$$-\frac{g^2}{4}\int\frac{\mathrm{d}^3p_1}{(2\pi)^3}\cdots\frac{\mathrm{d}^3p_4}{(2\pi)^3}\left(f^{ace}f^{bde}\frac{\delta_{ik}\delta_{jl}}{(\mathbf{p}_1+\mathbf{p}_3)^2}+f^{ade}f^{bce}\frac{\delta_{il}\delta_{jk}}{(\mathbf{p}_1+\mathbf{p}_4)^2}\right)$$
$$\times(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\,A_i^a(\mathbf{p}_1)A_j^b(\mathbf{p}_2)(2\pi)^3\frac{\delta}{\delta A_k^c(-\mathbf{p}_3)}(2\pi)^3\frac{\delta}{\delta A_l^d(-\mathbf{p}_4)}\,. \tag{3.14}$$

In the term (3.13) on the right-hand side, $t_{ij}(\mathbf{q})$ denotes the spatially transverse projector or transverse Kronecker delta

$$t_{ij}(\mathbf{q})\equiv\delta_{ij}-\hat{q}_i\hat{q}_j\,. \tag{3.15}$$

We shall now show, explicitly up to order $g^2$, that there is a unique perturbative solution of the Schrödinger equation (3.4) for the wave functional $\psi(\mathbf{A})$ in Eq. (3.3), if we only suppose that the dominant contribution to the coefficient function $f_k$ is at least of order $g^{k-2}$ for $k\geq 2$. A similar method for the determination of the vacuum wave functional has been applied before in Refs. [24–26] to a scalar theory and to Yang–Mills theory in Weyl gauge.

We will consider the case *without* external charges to begin with, and include charges $\rho_q$ later on in the context of the static potential. To order $g^0$, the Schrödinger equation reads

$$(N_\mathrm{c}^2-1)\left(\int\frac{\mathrm{d}^3p}{(2\pi)^3}f_2(\mathbf{p})\right)(2\pi)^3\delta(\mathbf{0})+\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}A_i^a(-\mathbf{p})\left[\mathbf{p}^2-\big(f_2(\mathbf{p})\big)^2\right]A_i^a(\mathbf{p})=E_0, \tag{3.16}$$

where we have used that

$$f_{2;ij}^{ab}(\mathbf{p},-\mathbf{p})=f_2(\mathbf{p})\delta_{ij}\delta^{ab}=f_2(-\mathbf{p})\delta_{ij}\delta^{ab} \tag{3.17}$$

(to be contracted with spatially transverse fields) as a consequence of the symmetry under the exchange of the arguments, of spatially rotational and global gauge symmetry, and of the fact that $f^{ab}_{2;ij}(\mathbf{p}_1, \mathbf{p}_2)$ is only defined for $\mathbf{p}_1 + \mathbf{p}_2 = 0$. Equation (3.16) implies that, to the current order,

$$f_2(\mathbf{p}) = |\mathbf{p}|, \tag{3.18}$$

$$E_0 = (N_c^2 - 1)\left(\int \frac{\mathrm{d}^3 p}{(2\pi)^3}|\mathbf{p}|\right)(2\pi)^3\delta(\mathbf{0}). \tag{3.19}$$

Generally, the energy $E_0$ cancels any field-independent terms multiplying the vacuum functional in the Schrödinger equation to any order in $g$. Eqs. (3.18) and (3.19) represent nothing but the well-known solution of the free ($g = 0$) theory. The choice of the sign in Eq. (3.18) is dictated by the normalizability of the wave functional (3.3) to order $g^0$. As usual, $(2\pi)^3\delta(\mathbf{0})$ is to be understood as the total volume of space.

To the next (first) order of $g$, the Schrödinger equation is not much more complicated: it reads

$$\begin{aligned}\frac{1}{3!}\int \frac{\mathrm{d}^3 p_1}{(2\pi)^3}\frac{\mathrm{d}^3 p_2}{(2\pi)^3}\frac{\mathrm{d}^3 p_3}{(2\pi)^3}\Big\{&\mathrm{i}g f^{abc}\left[\delta_{jk}(p_{1,l} - p_{2,l}) + \delta_{kl}(p_{2,j} - p_{3,j}) + \delta_{lj}(p_{3,k} - p_{1,k})\right] \\ &- 3|\mathbf{p}_1| f^{abc}_{3;jkl}(-\mathbf{p}_1, -\mathbf{p}_2, -\mathbf{p}_3)\Big\} A^a_j(\mathbf{p}_1) A^b_k(\mathbf{p}_2) A^c_l(\mathbf{p}_3)(2\pi)^3\delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) = 0,\end{aligned} \tag{3.20}$$

where we have already taken into account the results (3.18), (3.19) and the fact that

$$f^{abc}_{3;ijk}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = f^{abc} f_{3;ijk}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3), \tag{3.21}$$

which is a consequence of global gauge symmetry and the invariance of the vacuum wave functional under charge conjugation. The unique solution of Eq. (3.20) with the full symmetry under the exchange of the arguments is

$$f^{abc}_{3;ijk}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \frac{-\mathrm{i}g f^{abc}}{|\mathbf{p}_1| + |\mathbf{p}_2| + |\mathbf{p}_3|}\left[\delta_{ij}(p_{1,k} - p_{2,k}) + \delta_{jk}(p_{2,i} - p_{3,i}) + \delta_{ki}(p_{3,j} - p_{1,j})\right]. \tag{3.22}$$

This equality, and all the following equalities with explicit spatial (Lorentz) indices, are proper equalities only after contracting with the corresponding number of transverse vector fields $\mathbf{A}$, or, equivalently, after contracting every external spatial index with a transverse projector, for example in Eq. (3.22) the index $i$ with $t_{il}(\mathbf{p}_1)$.

We will now consider the Schrödinger equation to order $g^2$. On the left-hand side, terms with four and two powers of $\mathbf{A}$ appear, which have to cancel separately, and an $\mathbf{A}$-independent term which must equal $E_0$ to this order. We begin with the term with four powers of $\mathbf{A}$. The quartic coupling (3.11) in the Hamiltonian has to be cancelled by terms stemming from the second functional derivative in Eq. (3.8) acting on the vacuum wave functional, and a contribution from Eq. (3.14). To order $g^2$, the coefficient functions $f_2$ of Eq. (3.18) and $f_3$ of Eq. (3.22) contribute, as well as the function $f_4$ which we will determine. As a result, the coefficient function $f_4$ in the vacuum wave functional takes

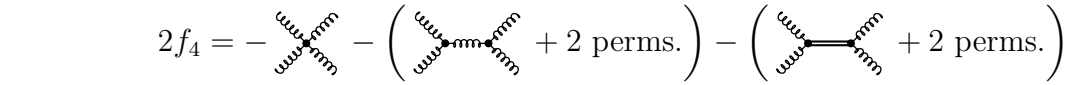

Figure 3.1: A diagrammatic representation of Eqs. (3.23)–(3.25). Every diagram corresponds to precisely one of the Eqs. (3.23)–(3.25), in the same order. The '2 perms.' refer to permutations of the external legs.

the following (fully symmetric) form to order $g^2$:

$$\begin{aligned}
&\big(|\mathbf{p}_1|+\ldots+|\mathbf{p}_4|\big)\,f^{abcd}_{4;ijkl}(\mathbf{p}_1,\ldots,\mathbf{p}_4) = \\
&\quad = g^2\Big[f^{abe}f^{cde}(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}) \\
&\qquad\qquad + f^{ace}f^{bde}(\delta_{ij}\delta_{kl}-\delta_{il}\delta_{jk}) \\
&\qquad\qquad\qquad + f^{ade}f^{bce}(\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl})\Big] \qquad (3.23)\\
&\qquad - \Big[f^{abe}_{3;ijm}(\mathbf{p}_1,\mathbf{p}_2,-\mathbf{p}_1-\mathbf{p}_2)t_{mn}(\mathbf{p}_1+\mathbf{p}_2)f^{cde}_{3;kln}(\mathbf{p}_3,\mathbf{p}_4,\mathbf{p}_1+\mathbf{p}_2) \\
&\qquad\quad + f^{ace}_{3;ikm}(\mathbf{p}_1,\mathbf{p}_3,-\mathbf{p}_1-\mathbf{p}_3)t_{mn}(\mathbf{p}_1+\mathbf{p}_3)f^{bde}_{3;jln}(\mathbf{p}_2,\mathbf{p}_4,\mathbf{p}_1+\mathbf{p}_3) \\
&\qquad\quad + f^{ade}_{3;ilm}(\mathbf{p}_1,\mathbf{p}_4,-\mathbf{p}_1-\mathbf{p}_4)t_{mn}(\mathbf{p}_1+\mathbf{p}_4)f^{bce}_{3;jkn}(\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_1+\mathbf{p}_4)\Big] \qquad (3.24)\\
&\quad -g^2\Big\{f^{abe}f^{cde}\,\delta_{ij}\delta_{kl}\,\frac{(|\mathbf{p}_1|-|\mathbf{p}_2|)(|\mathbf{p}_3|-|\mathbf{p}_4|)}{(\mathbf{p}_1+\mathbf{p}_2)^2} \\
&\qquad\qquad + f^{ace}f^{bde}\,\delta_{ik}\delta_{jl}\,\frac{(|\mathbf{p}_1|-|\mathbf{p}_3|)(|\mathbf{p}_2|-|\mathbf{p}_4|)}{(\mathbf{p}_1+\mathbf{p}_3)^2} \\
&\qquad\qquad\qquad + f^{ade}f^{bce}\,\delta_{il}\delta_{jk}\,\frac{(|\mathbf{p}_1|-|\mathbf{p}_4|)(|\mathbf{p}_2|-|\mathbf{p}_3|)}{(\mathbf{p}_1+\mathbf{p}_4)^2}\Big\}. \qquad (3.25)
\end{aligned}$$

This result for $f_4$ is represented diagrammatically in Fig. 3.1. Equation (3.23), divided by $(|\mathbf{p}_1|+\ldots+|\mathbf{p}_4|)$, is interpreted as the elementary or "bare" four-gluon vertex. The role of the factor 2 and the signs in Fig. 3.1 will become clear in the next section. Equation (3.24) and the second diagram in Fig. 3.1 represent the contraction of two elementary three-gluon vertices, the latter being given mathematically by Eq. (3.22). The contraction refers to spatial and colour indices and the momenta, with opposite signs. Note that there is no "propagator" factor associated with the contraction (except for a transverse Kronecker delta), and there is a factor $1/(|\mathbf{p}_1|+\ldots+|\mathbf{p}_4|)$ for the external momenta which is unusual from a diagrammatic point of view. Finally, Eq. (3.25) and the last diagram in Fig. 3.1 describe an "elementary" Coulomb interaction between the external gluon lines.

With this result in hand, we can go on to consider the terms quadratic in $\mathbf{A}$ in the Schrödinger equation to order $g^2$. The relevant contributions originate from Eqs. (3.8), (3.9), (3.13), and (3.14), and involve the functions $f_2$ and $f_4$. We obtain the following equation for the coefficient function $f_2$ to order $g^2$:

$$\begin{aligned}
\big(f_2(\mathbf{p})\big)^2\delta^{ab}\delta_{ij} &= \left(\mathbf{p}^2-\frac{N_\mathrm{c}g^2}{2}\,|\mathbf{p}|\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2}{(\mathbf{p}-\mathbf{q})^2}\right)\delta^{ab}\delta_{ij} \\
&\quad + \frac{1}{2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,f^{abcc}_{4;ijkl}(-\mathbf{p},\mathbf{p},-\mathbf{q},\mathbf{q})\,t_{kl}(\mathbf{q}) - N_\mathrm{c}g^2\delta^{ab}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{|\mathbf{p}|-|\mathbf{q}|}{(\mathbf{p}-\mathbf{q})^2}\,t_{ij}(\mathbf{q}). \qquad (3.26)
\end{aligned}$$

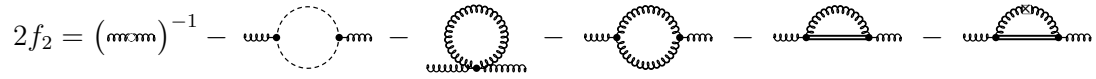

Figure 3.2: The diagrams corresponding to Eqs. (3.27)–(3.30). The bare propagator, the inverse $2|\mathbf{p}|$ of which appears in Eq. (3.27), is marked with an open circle for later use. The first two one-loop diagrams correspond to the integrals in Eq. (3.27), in the same order. The following diagrams represent Eqs. (3.28)–(3.30), respectively. See the text for a motivation of the "crossed" gluon propagator notation in the last loop diagram.

The explicit expression for $f_2$ to order $g^2$ is

$$f_2(\mathbf{p}) = |\mathbf{p}| - \frac{N_\mathrm{c} g^2}{4} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1 - (\hat{\mathbf{p}} \cdot \hat{\mathbf{q}})^2}{(\mathbf{p} - \mathbf{q})^2} + \frac{N_\mathrm{c} g^2}{2|\mathbf{p}|} \frac{4}{3} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{2|\mathbf{p}| + 2|\mathbf{q}|} \tag{3.27}$$

$$- \frac{N_\mathrm{c} g^2}{2|\mathbf{p}|} 2 \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\left(\delta_{ik} p_l + \delta_{kl} q_i - \delta_{li} p_k\right) t_{km}(\mathbf{p} - \mathbf{q})\, t_{ln}(\mathbf{q})}{2|\mathbf{p}| + 2|\mathbf{q}|}$$

$$\times \frac{\left(\delta_{jm} p_n + \delta_{mn} q_j - \delta_{nj} p_m\right) t_{ij}(\mathbf{p})}{(|\mathbf{p}| + |\mathbf{q}| + |\mathbf{p} - \mathbf{q}|)^2} \tag{3.28}$$

$$- \frac{N_\mathrm{c} g^2}{2|\mathbf{p}|} \frac{1}{2} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1 + (\hat{\mathbf{p}} \cdot \hat{\mathbf{q}})^2}{2|\mathbf{p}| + 2|\mathbf{q}|} \frac{\left(|\mathbf{p}| - |\mathbf{q}|\right)^2}{(\mathbf{p} - \mathbf{q})^2} \tag{3.29}$$

$$- \frac{N_\mathrm{c} g^2}{2|\mathbf{p}|} \frac{1}{2} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \left(1 + (\hat{\mathbf{p}} \cdot \hat{\mathbf{q}})^2\right) \frac{|\mathbf{p}| - |\mathbf{q}|}{(\mathbf{p} - \mathbf{q})^2}\,, \tag{3.30}$$

where we have used the contraction of an arbitrary tensor $T_{ij}(\mathbf{p})$

$$\frac{1}{2} T_{ij}(\mathbf{p}) t_{ij}(\mathbf{p}) = T^t(\mathbf{p}) \tag{3.31}$$

in order to extract the transverse part. We have presented the diagrams corresponding to Eqs. (3.27)–(3.30) in Fig. 3.2. The first loop integral in Eqs. (3.26) and (3.27) results from Eq. (3.9) and is represented in Fig. 3.2 as a ghost loop because it stems from the Faddeev–Popov determinant. The following three loop diagrams in Fig. 3.2 are obtained by contracting two external legs in the diagrams of Fig. 3.1, see Eq. (3.26). The last loop integral in Eq. (3.26), or the integral (3.30), on the other hand, originates from the terms (3.13) and (3.14) in the Hamiltonian. Lacking a better notation, we distinguish this contribution from the contraction of Eq. (3.25), the previous diagram, by marking the gluon propagator with a cross (because there is no term $|\mathbf{q}|$ in the denominator that would indicate the presence of an internal gluon propagator—in fact, the diagram may be interpreted to contain a $\Pi\Pi$-correlator, where $\mathbf{\Pi}$ is the momentum conjugate to $\mathbf{A}$).

We have thus completed the determination of the (exponent of the) perturbative vacuum wave functional to order $g^2$. The result is given in Eqs. (3.22), (3.23)–(3.25), and (3.27)–(3.30), to be substituted in Eq. (3.3). We can also extract the perturbative vacuum energy to the same order from the $\mathbf{A}$-independent terms in the Schrödinger equation with the result

$$E_0 = (N_\mathrm{c}^2 - 1) \left( \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_2(\mathbf{p}) \right) (2\pi)^3 \delta(\mathbf{0}) \tag{3.32}$$

[cf. Eq. (3.16)], where Eqs. (3.27)–(3.30) have to be substituted for $f_2(\mathbf{p})$. The explicit expression is not relevant for our purposes. We shall come back to the vacuum energy

later in the context of the static potential in the presence of external charges. It should also be clear by now how to take the determination of the perturbative vacuum functional and the vacuum energy systematically to higher orders.

## 3.3 Equal-time two-point correlation functions

For the calculation of the equal-time correlation functions, we need to include the Faddeev–Popov determinant in the measure of the functional integral, see Eq. (3.1). For our diagrammatic procedure, it is very convenient to introduce ghost fields and write

$$\mathsf{FP}(\mathbf{A}) = \int D[c,\bar{c}] \exp\left( - \int \mathrm{d}^3x\, \bar{c}^a(\mathbf{x})[-\boldsymbol{\nabla}\cdot\mathbf{D}^{ab}(\mathbf{A})]c^b(\mathbf{x}) \right). \tag{3.33}$$

In our conventions, we have explicitly

$$\int \mathrm{d}^3x\, \bar{c}^a(\mathbf{x})[-\boldsymbol{\nabla}\cdot\mathbf{D}^{ab}(\mathbf{A})]c^b(\mathbf{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\, \bar{c}^a(-\mathbf{p})\, \mathbf{p}^2 c^a(\mathbf{p}) \tag{3.34}$$

$$+ g \int \frac{\mathrm{d}^3p_1}{(2\pi)^3} \frac{\mathrm{d}^3p_2}{(2\pi)^3} \frac{\mathrm{d}^3p_3}{(2\pi)^3}\, \mathrm{i}f^{abc}\, p_{1,j}\, \bar{c}^a(\mathbf{p}_1) c^b(\mathbf{p}_2) A_j^c(\mathbf{p}_3) (2\pi)^3 \delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3). \tag{3.35}$$

Note that $p_{1,j}$ under the integral in Eq. (3.35) can be replaced by $-p_{2,j}$ due to the transversality of $\mathbf{A}$.

We now have a representation of the equal-time correlation functions as a functional integral over the transverse components of $\mathbf{A}$, the ghost and the antighost fields, see Eq. (3.1). The integration measure, which would be the exponential of the negative of the Euclidean action in the usual four-dimensional formulation (in Euclidean space), is now given by the exponential in Eq. (3.33) and the square of Eq. (3.3). Note that the vacuum functional is real (at least to order $g^2$) because the coefficient functions fulfil the reality condition

$$\left( f^{a_1\ldots a_k}_{k;i_1\ldots i_k}(-\mathbf{p}_1,\ldots,-\mathbf{p}_k) \right)^* = f^{a_1\ldots a_k}_{k;i_1\ldots i_k}(\mathbf{p}_1,\ldots,\mathbf{p}_k). \tag{3.36}$$

We shall use the analogy of this representation with the familiar functional integral representation of the covariant correlation functions in the usual four-dimensional formulation for the perturbative determination of the equal-time correlation functions (3.1). The corresponding Feynman rules are easily identified: the (static) gluon propagator is the inverse of $2|\mathbf{p}|$, cf. Eq. (3.18) (the factor of two is due to the square of the wave functional in the measure), the other contributions $-2\big(f_2(\mathbf{p}) - |\mathbf{p}|\big)$ and the other coefficient functions $-2f_3(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ and $-2f_4(\mathbf{p}_1,\ldots,\mathbf{p}_4)$ determine the two-, three-, and four-gluon vertices. Furthermore, from Eq. (3.34) we identify the free ghost propagator $1/\mathbf{p}^2$ and from Eq. (3.35) the ghost-gluon vertex. For the case of $\phi^4$ theory in $(1+1)$ dimensions, the calculation of equal-time correlation functions from a representation analogous to Eq. (3.1) has been discussed in Ref. [27].

We consider the gluon equal-time two-point function $\langle A_i^a(\mathbf{p}_1) A_j^b(\mathbf{p}_2)\rangle$ (with $t=0$ in the arguments of the gluon fields to be understood) first. One of the contributions to be taken into account is the ghost loop, constructed from two ghost-gluon vertices (3.35) and two ghost propagators [see Eq. (3.34)], and furthermore two static gluon propagators from Eq. (3.18) for the external lines. As it turns out, this contribution is exactly cancelled by the other contribution with the same graph "topology" which arises from contracting one of

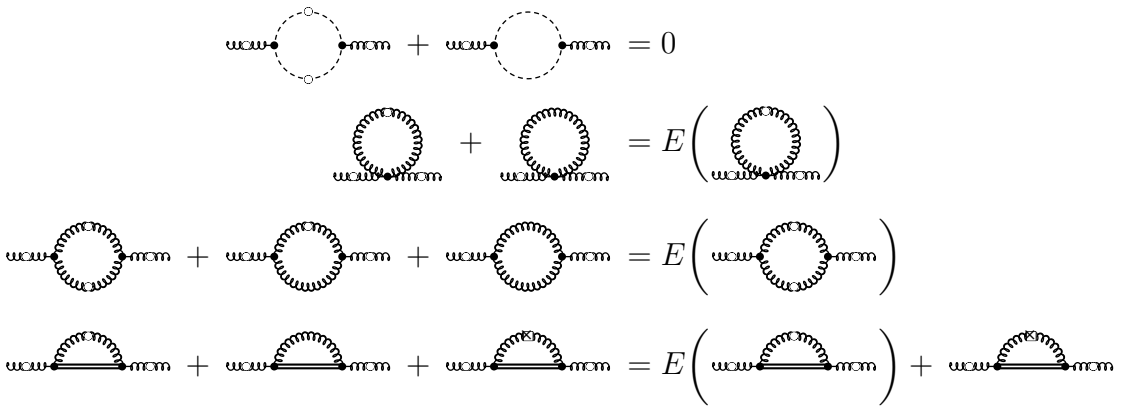

Figure 3.3: Diagrammatic representation of the various contributions to the gluonic equal-time two-point function, see Eqs. (3.37)–(3.41). The propagators marked with open circles are taken from Eqs. (3.18) and (3.34), respectively, while the "direct" contractions without open circles refer to the contractions that appear in the course of the determination of the vacuum wave functional, see Figs. 3.1 and 3.2, so that the corresponding parts of the diagrams translate into (minus two times) the mathematical expressions (3.24)–(3.25) [divided by $(|\mathbf{p}_1| + \ldots + |\mathbf{p}_4|)$] and (3.27)–(3.30). The notation '$E(\cdot)$' for the sum of all diagrams with the same topology is explained in the text, following Eq. (3.44).

the two-gluon vertices, (minus twice) the first integral in Eq. (3.27), with two external gluon propagators. Both contributions are represented diagrammatically in the first line of Fig. 3.3. The cancellation of the ghost loop contribution from the perturbative gluon two-point function (to one-loop order) is interesting given that this contribution plays a major role in the non-perturbative approaches to the infrared behavior of the equal-time gluon two-point function [2–7]. The cancellation of ghost loops was found to be a general feature in the Lagrangian functional integral approach [18,20]. An alternative way to see the cancellation in our present approach is to write the Faddeev–Popov determinant as

$$\mathsf{FP}(\mathbf{A}) = \exp\bigl[\mathrm{Tr}\ln\bigl(-\boldsymbol{\nabla}\cdot\mathbf{D}(\mathbf{A})\bigr)\bigr]. \tag{3.37}$$

The coefficient of the term quadratic in $\mathbf{A}$ in $\mathrm{Tr}\ln\bigl(-\boldsymbol{\nabla}\cdot\mathbf{D}(\mathbf{A})\bigr)$ precisely equals twice the first integral in Eq. (3.27) and hence cancels out in the exponent.

Next we turn to the tadpole contribution which is obtained from the elementary four-gluon vertex extracted from Eq. (3.23) appropriately contracted with three static gluon propagators. Again, there is a second contribution with the same "topology" given by the two-gluon vertex from the last integral in Eq. (3.27) contracted with two external propagators, cf. the second line in Fig. 3.3. The sum of these two contributions to the gluon equal-time two-point function is (with $\mathbf{p} \equiv \mathbf{p}_1$)

$$-\frac{2N_{\mathrm{c}}g^2}{(2|\mathbf{p}|)^2}\,\frac{4}{3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1}{2|\mathbf{p}|+2|\mathbf{q}|}\,\frac{1}{2|\mathbf{q}|} - \frac{2N_{\mathrm{c}}g^2}{(2|\mathbf{p}|)^3}\,\frac{4}{3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1}{2|\mathbf{p}|+2|\mathbf{q}|}$$
$$= -\frac{2N_{\mathrm{c}}g^2}{(2|\mathbf{p}|)^3}\,\frac{4}{3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1}{2|\mathbf{q}|}\,, \tag{3.38}$$

to be multiplied with $\delta^{ab}t_{ij}(\mathbf{p})$.

The most complicated contribution to the two-point function comes from diagrams with the gluon loop topology (with two three-gluon vertices). There are three different diagrams of this type, represented in the third line of Fig. 3.3, the first from contracting two three-gluon vertices extracted from Eq. (3.22) with four static gluon propagators (two internal and two external), the second from contracting the part of the four-gluon vertex given by Eq. (3.24) with one internal and two external gluon propagators, and the third by contracting the two-gluon vertex from Eq. (3.28) with two static gluon propagators. The sum of these contributions is

$$\frac{2N_c g^2}{(2|\mathbf{p}|)^3}\,2\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{2|\mathbf{p}|+2|\mathbf{p}-\mathbf{q}|}{(|\mathbf{p}|+|\mathbf{q}|+|\mathbf{p}-\mathbf{q}|)^2\,2|\mathbf{q}|\,2|\mathbf{p}-\mathbf{q}|} \times\left(\delta_{km}p_n+\delta_{mn}q_k-\delta_{nk}p_m\right)t_{mr}(\mathbf{p}-\mathbf{q})\,t_{ns}(\mathbf{q})\left(\delta_{lr}p_s+\delta_{rs}q_l-\delta_{sl}p_r\right)t_{kl}(\mathbf{p}) \quad (3.39)$$

(again, to be multiplied with $\delta^{ab}t_{ij}(\mathbf{p})$, and $\mathbf{p}\equiv\mathbf{p}_1$). The tensor structure in this expression is invariant under the transformation $\mathbf{q}\to\mathbf{p}-\mathbf{q}$, a fact we can use to replace $2|\mathbf{p}|+2|\mathbf{p}-\mathbf{q}|$ in the numerator with $2|\mathbf{p}|+|\mathbf{q}|+|\mathbf{p}-\mathbf{q}|$. The tensor structure itself can be simplified by performing the contractions explicitly. A straightforward, but somewhat tedious calculation gives

$$\left(\delta_{km}p_n+\delta_{mn}q_k-\delta_{nk}p_m\right)t_{mr}(\mathbf{p}-\mathbf{q})\,t_{ns}(\mathbf{q})\left(\delta_{lr}p_s+\delta_{rs}q_l-\delta_{sl}p_r\right)t_{kl}(\mathbf{p}) = = \left(1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2\right)\left(2\mathbf{p}^2+2\mathbf{q}^2+\frac{\mathbf{p}^2\,\mathbf{q}^2+(\mathbf{p}\cdot\mathbf{q})^2}{(\mathbf{p}-\mathbf{q})^2}\right). \quad (3.40)$$

Finally, we turn to the contributions that involve the (non-abelian) Coulomb potential. There are, again, three such terms, represented in the last line of Fig. 3.3, the first from the four-gluon vertex derived from Eq. (3.25) contracted with three static gluon propagators, and the other two using the two-gluon vertices corresponding to the two integrals (3.29) and (3.30) contracted with two gluon propagators each. The sum of these terms is

$$\frac{2N_c g^2}{(2|\mathbf{p}|)^3}\,\frac{1}{2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1+(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2}{2|\mathbf{q}|}\,\frac{(|\mathbf{p}|-|\mathbf{q}|)^2}{(\mathbf{p}-\mathbf{q})^2}+ + \frac{2N_c g^2}{(2|\mathbf{p}|)^3}\,\frac{1}{2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\left(1+(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2\right)\frac{|\mathbf{p}|-|\mathbf{q}|}{(\mathbf{p}-\mathbf{q})^2} = = \frac{2N_c g^2}{(2|\mathbf{p}|)^3}\,\frac{1}{2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1+(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2}{2|\mathbf{q}|}\,\frac{|\mathbf{p}|^2-|\mathbf{q}|^2}{(\mathbf{p}-\mathbf{q})^2}, \quad (3.41)$$

to be multiplied with $\delta^{ab}t_{ij}(\mathbf{p})$ as before, and $\mathbf{p}\equiv\mathbf{p}_1$. On the left-hand side of Eq. (3.41), we have added up the contributions from the contraction of the four-gluon vertex and the two-gluon vertex in Eq. (3.29) to give the first loop integral. The left-hand side of Eq. (3.41) is represented diagrammatically as the right-hand side of the last line in Fig. 3.3.

Putting it all together, the result for the equal-time gluon two-point function is, to order $g^2$,

$$\langle A_i^a(\mathbf{p}_1)A_j^b(\mathbf{p}_2)\rangle = \left[\frac{1}{2|\mathbf{p}_1|}-\frac{2N_c g^2}{(2|\mathbf{p}_1|)^3}\,\frac{4}{3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1}{2|\mathbf{q}|}\right. + \left.\frac{2N_c g^2}{(2|\mathbf{p}_1|)^3}\,2\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{\left(1-(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{q}})^2\right)(2|\mathbf{p}_1|+|\mathbf{q}|+|\mathbf{p}_1-\mathbf{q}|)}{(|\mathbf{p}_1|+|\mathbf{q}|+|\mathbf{p}_1-\mathbf{q}|)^2\,2|\mathbf{q}|\,2|\mathbf{p}_1-\mathbf{q}|}\right] \quad (3.42)$$

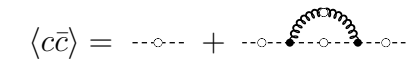

Figure 3.4: Diagrammatic representation of Eq. (3.45).

$$\times \left( 2\mathbf{p}_1^2 + 2\mathbf{q}^2 + \frac{\mathbf{p}_1^2\,\mathbf{q}^2 + (\mathbf{p}_1\cdot\mathbf{q})^2}{(\mathbf{p}_1-\mathbf{q})^2} \right) \tag{3.43}$$

$$+ \frac{2N_c g^2}{(2|\mathbf{p}_1|)^3}\,\frac{1}{2}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\,\frac{1+(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{q}})^2}{2|\mathbf{q}|}\,\frac{\mathbf{p}_1^2-\mathbf{q}^2}{(\mathbf{p}_1-\mathbf{q})^2}\Bigg]\,\delta^{ab}\,t_{ij}(\mathbf{p}_1)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2). \tag{3.44}$$

Looking at the diagrams on the left-hand sides of the first three lines in Fig. 3.3, it is clear that there is precisely one diagram among those of the same topology that is contructed exclusively from the elementary vertices (3.22), (3.23), and (3.35) and the "bare" propagators taken from Eqs. (3.18) and (3.34). We will call this kind of diagram an 'F-diagram'. Interestingly, the sum of all diagrams with the same topology which we will refer to as an 'E-diagram', can be constructed from the corresponding F-diagram by a formal operation that we call the 'E-operator'. It is denoted as '$E(\cdot)$' in Fig. 3.3 and will be illustrated considering the example of the third line in Fig. 3.3: starting from the mathematical expression for the corresponding F-diagram, one multiplies the integrand (of the integral over loop momentum) with the sum of all $\mathbf{k}$ where $\mathbf{k}$ runs over the momenta of all propagators in the diagram, and divides by the corresponding sum restricted to the momenta of the external propagators. Indeed, the result of this operation is given by Eq. (3.39) in the $(\mathbf{q}\to\mathbf{p}-\mathbf{q})$-symmetric form, see the remark after Eq. (3.39).

The same rule applies to the second line in Fig. 3.3, or Eq. (3.38), only that the propagator that starts and ends at the same vertex has to be counted twice in the sum over (internal and external) $|\mathbf{k}|$. Similarly, the E-operator can be used to sum the first two diagrams on the left-hand side of the last line in Fig. 3.3. We have to consider the Coulomb interaction as an elementary vertex given by Eq. (3.25) to this end, and count the gluon propagator that starts and ends at this vertex twice in the sum over $|\mathbf{k}|$ just as in the case of the other elementary four-gluon vertex. The same rule for the generation of the E-diagrams (given the elementary vertices and propagators) has been shown to hold up to two-loop order for the equal-time two-point function and to one-loop order for the four-point function in the context of a scalar $\phi^4$ theory [28] (a detailed account will be given elsewhere). Note that two contributions, the second diagram in the first line of Fig. 3.3 and the third diagram in the last line in the same figure corresponding to the first loop integral in Eq. (3.27) and to Eq. (3.30), respectively, do not fit into this general scheme.

The calculation of the equal-time ghost two-point function from the graphical rules is much simpler. One obtains directly

$$\langle c^a(\mathbf{p}_1)\bar{c}^b(\mathbf{p}_2)\rangle = \left(\frac{1}{\mathbf{p}_1^2} + \frac{N_c g^2}{\mathbf{p}_1^2}\int\frac{\mathrm{d}^3 q}{(2\pi)^3}\,\frac{1-(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{q}})^2}{(\mathbf{p}_1-\mathbf{q})^2\,2|\mathbf{q}|}\right)\delta^{ab}\,(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2), \tag{3.45}$$

corresponding to the diagrams in Fig. 3.4. Note that one of the factors $1/\mathbf{p}_1^2$ for the external ghost propagators cancels against the momentum dependence of the ghost-gluon vertices.

We shall close this section with a calculation of the static (heavy quark) potential, the energy for a configuration of static external colour charges $\rho_q(\mathbf{x})$. To this end, we introduce

charges $\rho_q(\mathbf{x})$ into the Hamiltonian, see Eqs. (3.7) and (3.12). Compared to the Coulomb term (3.13)–(3.14) which was calculated in the absence of external charges, there are two new terms of order $g^2$:

$$\frac{g^2}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\rho_q^a(-\mathbf{p})\,\frac{1}{\mathbf{p}^2}\,\rho_q^a(\mathbf{p}), \tag{3.46}$$

which is **A**-independent and hence only contributes to the vacuum energy but leaves the vacuum wave functional unchanged, and

$$g^2\int\frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{(2\pi)^3}\,\frac{f^{abc}}{\mathbf{p}_1^2}(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\,\rho_q^a(\mathbf{p}_1)A_j^b(\mathbf{p}_2)\,\frac{1}{\mathrm{i}}(2\pi)^3\frac{\delta}{\delta A_j^c(-\mathbf{p}_3)}\,. \tag{3.47}$$

The latter term, when applied to the vacuum wave functional, generates the following (properly symmetrized) expression to order $g^2$,

$$-\frac{g^2}{2}\int\frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{(2\pi)^3}\,\mathrm{i}f^{abc}\,\frac{|\mathbf{p}_2|-|\mathbf{p}_3|}{\mathbf{p}_1^2}\rho_q^a(\mathbf{p}_1)A_j^b(\mathbf{p}_2)A_j^c(\mathbf{p}_3)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3), \tag{3.48}$$

which implies that the vacuum wave functional to order $g^2$ has to be modified in order to fulfil the Schrödinger equation with this new term.

A term cancelling the expression (3.48) in the Schrödinger equation can only result from the second derivative term in the Hamiltonian [Eq. (3.8)]. It is then simple to see that we have to add the expression

$$-\frac{g^2}{2}\int\frac{\mathrm{d}^3p_1}{(2\pi)^3}\frac{\mathrm{d}^3p_2}{(2\pi)^3}\frac{\mathrm{d}^3p_3}{(2\pi)^3}\,\frac{\mathrm{i}f^{abc}}{\mathbf{p}_1^2}\,\frac{|\mathbf{p}_2|-|\mathbf{p}_3|}{|\mathbf{p}_2|+|\mathbf{p}_3|}\,\rho_q^a(\mathbf{p}_1)A_j^b(\mathbf{p}_2)A_j^c(\mathbf{p}_3)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \tag{3.49}$$

to the negative of the exponent of the vacuum wave functional in order to satisfy the Schrödinger equation to order $g^2$. This term describes the back-reaction of the vacuum to the presence of the external charges to order $g^2$. Observe that due to the presence of the external charges $\rho_q(\mathbf{p})$, the coefficient function of $A_i^a(\mathbf{p}_1)A_j^b(\mathbf{p}_2)$ in the vacuum wave functional ceases to be of the form $f_2(\mathbf{p}_2)\delta^{ab}\,\delta_{ij}(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2)$. Furthermore, contrary to the terms found before, the contribution (3.49) is imaginary. The rest of the vacuum wave functional determined in the previous section remains without change.

We now have to calculate the vacuum energy in the presence of the external charges. To order $g^2$, the result is the former one, Eq. (3.32), without any contribution from the new term (3.49), plus Eq. (3.46) which is the part of the energy that depends on the external charges and hence defines the potential to this order. Of course, this is just the well-known Coulomb potential of electrodynamics. What we are really interested in are the first quantum corrections to this "bare" potential, which are of order $g^4$.

In general, the vacuum energy is given by

$$E_0=\frac{g^2}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\rho_q^a(-\mathbf{p})\frac{1}{\mathbf{p}^2}\,\rho_q^a(\mathbf{p}) - \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}(2\pi)^3\frac{\delta}{\delta A_i^a(\mathbf{p})}(2\pi)^3\frac{\delta}{\delta A_i^a(-\mathbf{p})}\,\psi(\mathbf{A})\bigg|_{\mathbf{A}=0}, \tag{3.50}$$

which reduces to Eq. (3.32) in the absence of external charges. A contribution of order $g^4$ can hence only originate from the terms in the vacuum wave functional that are quadratic

in **A**. As long as we are only interested in the potential between static sources, we can concentrate on terms that contain precisely two powers of $\rho_q$. We then start by identifying all the contributions to the Schrödinger equation of order $g^4$ that contain two powers of **A** and two powers of $\rho_q$. One of these contributions results from expanding the Coulomb kernel

$$\langle \mathbf{x}, a | \, (-\boldsymbol{\nabla} \cdot \mathbf{D})^{-1} (-\nabla^2) (-\boldsymbol{\nabla} \cdot \mathbf{D})^{-1} \, | \mathbf{y}, b \rangle \tag{3.51}$$

in Eq. (3.5) to second order in **A** for $\rho = \rho_q$. The result is the term

$$-\frac{3}{4}\, g^4 \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 p_4}{(2\pi)^3} \left( f^{ace} f^{bde} \, \frac{p_{1,i} p_{2,j}}{\mathbf{p}_1^2 \, \mathbf{p}_2^2 \, (\mathbf{p}_1 + \mathbf{p}_3)^2} + f^{ade} f^{bce} \, \frac{p_{1,j} p_{2,i}}{\mathbf{p}_1^2 \, \mathbf{p}_2^2 \, (\mathbf{p}_1 + \mathbf{p}_4)^2} \right) \times \rho_q^a(\mathbf{p}_1) \rho_q^b(\mathbf{p}_2) A_i^c(\mathbf{p}_3) A_j^d(\mathbf{p}_4) (2\pi)^3 \delta(\mathbf{p}_1 + \ldots + \mathbf{p}_4) \tag{3.52}$$

on the left-hand side of the Schrödinger equation.

Another contribution of the same type arises from the second functional derivative in Eq. (3.8) acting (twice) on the term (3.49), which gives the contribution

$$\begin{aligned} \frac{g^4}{4} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 p_4}{(2\pi)^3} \frac{1}{\mathbf{p}_1^2 \, \mathbf{p}_2^2} \Bigg[ & f^{ace} f^{bde} \, t_{ij}(\mathbf{p}_1 + \mathbf{p}_3) \, \frac{|\mathbf{p}_1 + \mathbf{p}_3| - |\mathbf{p}_3|}{|\mathbf{p}_1 + \mathbf{p}_3| + |\mathbf{p}_3|} \, \frac{|\mathbf{p}_1 + \mathbf{p}_3| - |\mathbf{p}_4|}{|\mathbf{p}_1 + \mathbf{p}_3| + |\mathbf{p}_4|} \\ & + f^{ade} f^{bce} \, t_{ij}(\mathbf{p}_1 + \mathbf{p}_4) \, \frac{|\mathbf{p}_1 + \mathbf{p}_4| - |\mathbf{p}_3|}{|\mathbf{p}_1 + \mathbf{p}_4| + |\mathbf{p}_3|} \, \frac{|\mathbf{p}_1 + \mathbf{p}_4| - |\mathbf{p}_4|}{|\mathbf{p}_1 + \mathbf{p}_4| + |\mathbf{p}_4|} \Bigg] \\ & \times \rho_q^a(\mathbf{p}_1) \rho_q^b(\mathbf{p}_2) A_i^c(\mathbf{p}_3) A_j^d(\mathbf{p}_4) (2\pi)^3 \delta(\mathbf{p}_1 + \ldots + \mathbf{p}_4) \end{aligned} \tag{3.53}$$

to the Schrödinger equation. The last contribution of the same type comes from the "mixed" term where the operator (3.47) acts upon the expression (3.49) in the wave functional. The result is

$$\begin{aligned} -\frac{g^4}{4} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 p_4}{(2\pi)^3} \, & \frac{1}{\mathbf{p}_1^2 \, \mathbf{p}_2^2} \\ & \times \Bigg[ f^{ace} f^{bde} \, t_{ij}(\mathbf{p}_1 + \mathbf{p}_3) \left( \frac{|\mathbf{p}_1 + \mathbf{p}_3| - |\mathbf{p}_3|}{|\mathbf{p}_1 + \mathbf{p}_3| + |\mathbf{p}_3|} + \frac{|\mathbf{p}_1 + \mathbf{p}_3| - |\mathbf{p}_4|}{|\mathbf{p}_1 + \mathbf{p}_3| + |\mathbf{p}_4|} \right) \\ & \quad + f^{ade} f^{bce} \, t_{ij}(\mathbf{p}_1 + \mathbf{p}_4) \left( \frac{|\mathbf{p}_1 + \mathbf{p}_4| - |\mathbf{p}_3|}{|\mathbf{p}_1 + \mathbf{p}_4| + |\mathbf{p}_3|} + \frac{|\mathbf{p}_1 + \mathbf{p}_4| - |\mathbf{p}_4|}{|\mathbf{p}_1 + \mathbf{p}_4| + |\mathbf{p}_4|} \right) \Bigg] \\ & \times \rho_q^a(\mathbf{p}_1) \rho_q^b(\mathbf{p}_2) A_i^c(\mathbf{p}_3) A_j^d(\mathbf{p}_4) (2\pi)^3 \delta(\mathbf{p}_1 + \ldots + \mathbf{p}_4), \end{aligned} \tag{3.54}$$

to be included in the Schrödinger equation. It can be shown that no other contributions quadratic in **A** and in $\rho_q$ exist to order $g^4$.

In analogy to the determination of the expression (3.49) from Eq. (3.48), the three contributions (3.52)–(3.54) to the Schrödinger equation are taken care of by including the following expression in the negative exponent of the vacuum wave functional [in addition to (3.22), (3.23)–(3.25), (3.27)–(3.30), and (3.49)]

$$-\frac{g^4}{4} \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3} \cdots \frac{\mathrm{d}^3 p_4}{(2\pi)^3} \frac{1}{\mathbf{p}_1^2 \, \mathbf{p}_2^2 \, (|\mathbf{p}_3| + |\mathbf{p}_4|)}$$

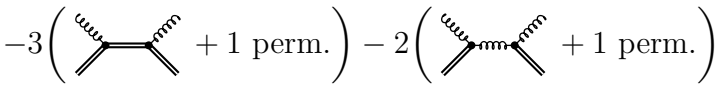

Figure 3.5: Diagrammatic representation of the contributions (3.55) (multiplied by 2) to the vacuum wave functional.

$$\begin{aligned}
&\Bigg\{ f^{ace} f^{bde} \Bigg[ \frac{3 p_{1,i} p_{2,j}}{(\mathbf{p}_1+\mathbf{p}_3)^2} - t_{ij}(\mathbf{p}_1+\mathbf{p}_3) \\
&\times \left( \frac{|\mathbf{p}_1+\mathbf{p}_3|-|\mathbf{p}_3|}{|\mathbf{p}_1+\mathbf{p}_3|+|\mathbf{p}_3|} \frac{|\mathbf{p}_1+\mathbf{p}_3|-|\mathbf{p}_4|}{|\mathbf{p}_1+\mathbf{p}_3|+|\mathbf{p}_4|} - \frac{|\mathbf{p}_1+\mathbf{p}_3|-|\mathbf{p}_3|}{|\mathbf{p}_1+\mathbf{p}_3|+|\mathbf{p}_3|} - \frac{|\mathbf{p}_1+\mathbf{p}_3|-|\mathbf{p}_4|}{|\mathbf{p}_1+\mathbf{p}_3|+|\mathbf{p}_4|} \right) \Bigg] \\
&+ f^{ade} f^{bce} \Bigg[ \frac{3 p_{1,j} p_{2,i}}{(\mathbf{p}_1+\mathbf{p}_4)^2} - t_{ij}(\mathbf{p}_1+\mathbf{p}_4) \Bigg( \frac{|\mathbf{p}_1+\mathbf{p}_4|-|\mathbf{p}_3|}{|\mathbf{p}_1+\mathbf{p}_4|+|\mathbf{p}_3|} \frac{|\mathbf{p}_1+\mathbf{p}_4|-|\mathbf{p}_4|}{|\mathbf{p}_1+\mathbf{p}_4|+|\mathbf{p}_4|} - \frac{|\mathbf{p}_1+\mathbf{p}_4|-|\mathbf{p}_3|}{|\mathbf{p}_1+\mathbf{p}_4|+|\mathbf{p}_3|} \\
&- \frac{|\mathbf{p}_1+\mathbf{p}_4|-|\mathbf{p}_4|}{|\mathbf{p}_1+\mathbf{p}_4|+|\mathbf{p}_4|} \Bigg) \Bigg] \Bigg\} \rho_q^a(\mathbf{p}_1) \rho_q^b(\mathbf{p}_2) A_i^c(\mathbf{p}_3) A_j^d(\mathbf{p}_4) (2\pi)^3 \delta(\mathbf{p}_1+\ldots+\mathbf{p}_4). \quad (3.55)
\end{aligned}$$

This result (multiplied by 2) is represented diagrammatically in Fig. 3.5, where we have denoted the 'Coulomb propagator' $1/\mathbf{p}^2$ as a double line. The first diagram (and its permutation) corresponds to the expression (3.52), while the second diagram (plus its permutation) corresponds to the sum of the expressions (3.53) and (3.54). From Eq. (3.50), we find the contribution to the vacuum energy

$$\begin{aligned}
&\frac{g^2}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \rho_q^a(-\mathbf{p}) \frac{1}{(\mathbf{p}^2)^2} \Bigg\{ \frac{N_\mathrm{c} g^2}{2} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{2|\mathbf{q}|} \Bigg[ 3 \frac{\mathbf{p}^2 - (\mathbf{p}\cdot\hat{\mathbf{q}})^2}{(\mathbf{p}-\mathbf{q})^2} \\
&+ t_{ij}(\mathbf{q}) t_{ij}(\mathbf{p}-\mathbf{q}) \left( \left( \frac{|\mathbf{p}-\mathbf{q}|-|\mathbf{q}|}{|\mathbf{p}-\mathbf{q}|+|\mathbf{q}|} \right)^2 - 2 \frac{|\mathbf{p}-\mathbf{q}|-|\mathbf{q}|}{|\mathbf{p}-\mathbf{q}|+|\mathbf{q}|} \right) + 3 \frac{\mathbf{p}^2 - (\mathbf{p}\cdot\hat{\mathbf{q}})^2}{(\mathbf{p}+\mathbf{q})^2} \\
&+ t_{ij}(\mathbf{q}) t_{ij}(\mathbf{p}+\mathbf{q}) \left( \left( \frac{|\mathbf{p}+\mathbf{q}|-|\mathbf{q}|}{|\mathbf{p}+\mathbf{q}|+|\mathbf{q}|} \right)^2 - 2 \frac{|\mathbf{p}+\mathbf{q}|-|\mathbf{q}|}{|\mathbf{p}+\mathbf{q}|+|\mathbf{q}|} \right) \Bigg] \Bigg\} \rho_q^a(\mathbf{p}). \quad (3.56)
\end{aligned}$$

This latter expression can be simplified by shifting $\mathbf{q} \to \mathbf{q} - \mathbf{p}$ in the last two terms (in the round bracket). Together with Eq. (3.46), we find for the part of the vacuum energy that is quadratic in the external static charge $\rho_q$,

$$E_0^{(\rho_q,2)} = \frac{g^2}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \rho_q^a(-\mathbf{p}) V(\mathbf{p}) \rho_q^a(\mathbf{p}), \quad (3.57)$$

the following result for the static potential to order $g^2$

$$V(\mathbf{p}) = \frac{1}{\mathbf{p}^2} + \frac{N_\mathrm{c} g^2}{\mathbf{p}^2} 3 \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1 - (\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2}{2|\mathbf{q}|\,(\mathbf{p}-\mathbf{q})^2} \quad (3.58)$$

$$- \frac{N_\mathrm{c} g^2}{(\mathbf{p}^2)^2} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \left( 1 + \frac{\big((\mathbf{p}-\mathbf{q})\cdot\mathbf{q}\big)^2}{(\mathbf{p}-\mathbf{q})^2\,\mathbf{q}^2} \right) \frac{(|\mathbf{p}-\mathbf{q}|-|\mathbf{q}|)^2}{2|\mathbf{p}-\mathbf{q}|\,2|\mathbf{q}|\,(|\mathbf{p}-\mathbf{q}|+|\mathbf{q}|)}. \quad (3.59)$$

Note that to the proper colour Coulomb potential (see e.g. Ref. [31]), only the antiscreening term, the integral in Eq. (3.58), contributes, while the full static potential also contains the screening contribution (3.59). The semi-analytical variational approaches [3–6,19] have only considered the proper colour Coulomb potential so far.

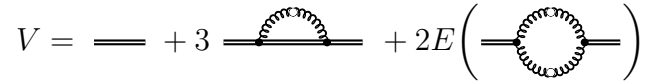

Figure 3.6: A diagrammatic interpretation of the static potential to order $g^2$.

We can associate diagrams with the different contributions in Eqs. (3.58)–(3.59) in a natural way. The vertex that joins two Coulomb (double) lines and one gluon line corresponds to the same mathematical expression as the ghost-gluon vertex since both objects originate from the Faddeev–Popov operator $(-\boldsymbol{\nabla}\cdot\mathbf{D})$ (or its inverse). On the other hand, the vertex with two gluon lines and one Coulomb line translates to the expression

$$\mathrm{i}gf^{abc}\,\frac{|\mathbf{p}_2|-|\mathbf{p}_3|}{|\mathbf{p}_2|+|\mathbf{p}_3|}\,\delta_{jk}\,, \tag{3.60}$$

where the gluon lines carry the momenta $\mathbf{p}_2$ and $\mathbf{p}_3$, (spatial) Lorentz indices $j$ and $k$, and colour indices $b$ and $c$ [cf. Eq. (3.49)]. Note that the "elementary" Coulomb interaction (3.25) is *different* from the contraction of two such vertices with a Coulomb propagator. With these conventions, we can represent the static potential as in Fig. 3.6. The E-operator in Fig. 3.6 exclusively refers to the internal gluon propagators and thus amounts to multiplying the integrand with $|\mathbf{p}-\mathbf{q}|+|\mathbf{q}|$.

We hence have succeeded in calculating the equal-time gluon and ghost two-point functions and the static potential to one-loop order in our functional perturbative approach, with the results (3.42)–(3.45) and (3.58)–(3.59). The same results can be obtained from a straightforward application of Rayleigh–Schrödinger perturbation theory [29]. Compared to these latter calculations, we have here developed a functional integral and diagrammatical approach that is potentially advantageous in higher-order perturbative calculations. We have also described a set of simplified diagrammatic rules (the 'E-operator') for the determination of equal-time correlation functions that is expected to carry over to higher perturbative orders and is hoped to eventually lead to non-perturbative equations for the equal-time correlation functions analogous to Dyson–Schwinger equations.

## 3.4 Lagrangian approach and renormalization

Naive power counting shows that the results of the preceding section, Eqs. (3.42)–(3.45) and (3.58)–(3.59), are ultraviolet (UV) divergent and need to be renormalized. However, some of the denominators occuring in the loop integrals are of a different type from those that usually appear in covariant perturbation theory, and efficient techniques for the handling of these terms have yet to be developed. These remarks apply in particular to Eqs. (3.43) and (3.59).

The equal-time correlation functions we have been calculating are a special or limiting case of the usual space-time correlation functions, whence we naturally obtain the representation

$$\langle A_i^a(\mathbf{p}_1,t=0)A_j^b(\mathbf{p}_2,t=0)\rangle=\int_{-\infty}^{\infty}\frac{\mathrm{d}p_{1,4}}{2\pi}\frac{\mathrm{d}p_{2,4}}{2\pi}\langle A_i^a(\mathbf{p}_1,p_{1,4})A_j^b(\mathbf{p}_2,p_{2,4})\rangle \tag{3.61}$$

(with the space-time correlation functions written in Euclidean space-time). Since the regularization and renormalization program has been developed for the space-time correlation functions, this representation is quite useful for our purposes. In the case of

Coulomb gauge Yang–Mills theory, however, covariance is explicitly broken through the gauge condition, and the calculation of the space-time correlation functions in the usual Lagrangian functional integral approach represents a difficulty by itself. Techniques have been developed to overcome these difficulties and applied in Ref. [20] to the calculation of the two-point correlation functions to one-loop order.

Before properly considering the renormalization of our results, we will verify that these coincide on a formal level with the expressions obtained via Eq. (3.61), taking for the space-time correlation functions the formulas derived in the Lagrangian functional integral approach in Ref. [20]. In our notation,

$$\langle A_i^a(p_1)A_j^b(p_2)\rangle = \left(\frac{1}{p_1^2} - \frac{N_{\rm c}g^2}{\left(p_1^2\right)^2}\,\frac{4}{3}\int\frac{{\rm d}^4q}{(2\pi)^4}\,\frac{1}{q^2}\right. \tag{3.62}$$

$$+\,\frac{N_{\rm c}g^2}{\left(p_1^2\right)^2}\int\frac{{\rm d}^4q}{(2\pi)^4}\,\frac{\left(\delta_{km}p_{1,n}+\delta_{mn}q_k-\delta_{nk}p_{1,m}\right)t_{mr}(\mathbf{p}_1-\mathbf{q})\,t_{ns}(\mathbf{q})}{q^2}$$

$$\times\,\frac{\left(\delta_{lr}p_{1,s}+\delta_{rs}q_l-\delta_{sl}p_{1,r}\right)t_{kl}(\mathbf{p}_1)}{(p_1-q)^2} \tag{3.63}$$

$$\left.+\,\frac{N_{\rm c}g^2}{\left(p_1^2\right)^2}\,\frac{1}{2}\int\frac{{\rm d}^4q}{(2\pi)^4}\,\frac{t_{kl}(\mathbf{p}_1)t_{kl}(\mathbf{q})\,(p_{1,4}^2-\mathbf{q}^2)}{q^2\,(\mathbf{p}_1-\mathbf{q})^2}\right)\delta^{ab}\,t_{ij}(\mathbf{p}_1)(2\pi)^4\delta(p_1+p_2), \tag{3.64}$$

with $p_1^2 \equiv \mathbf{p}_1^2 + p_{1,4}^2$ in Euclidean space-time. Formal integration of $p_{1,4}$ and $p_{2,4}$ as in Eq. (3.61) and of the component $q_4$ of the loop momentum, most easily using the residue theorem, leads to our equal-time correlation function (3.42)–(3.44). In Eq. (3.62), we have included the tadpole diagram in order that the correspondence with the equal-time gluon two-point function (3.42)–(3.44) be term by term. The tadpole diagram was not considered explicitly in Refs. [20,21] because it vanishes in dimensional regularization.

The case of the ghost two-point function is even simpler, because it is instantaneous already in the Lagrangian approach: from Ref. [20],

$$\langle c^a(p_1)\bar{c}^b(p_2)\rangle = \left(\frac{1}{\mathbf{p}_1^2} + \frac{N_{\rm c}g^2}{\left(\mathbf{p}_1^2\right)^2}\int\frac{{\rm d}^4q}{(2\pi)^4}\,\frac{p_{1,i}p_{1,j}\,t_{ij}(\mathbf{q})}{q^2\,(\mathbf{p}_1-\mathbf{q})^2}\right)\delta^{ab}\,(2\pi)^4\delta(p_1+p_2), \tag{3.65}$$

and the fact that the dependence on $p_{1,4}$ and $p_{2,4}$ is exclusively through the delta function for energy conservation implies that $\langle c^a(\mathbf{p}_1,t_1)\bar{c}^b(\mathbf{p}_2,t_2)\rangle$ contains the factor $\delta(t_1-t_2)$. Integrating over $p_{1,4}$ *and* $p_{2,4}$ or putting $t_1$ *and* $t_2$ to zero then results in a factor $\delta(0)$. In this case, in order to reproduce Eq. (3.45), we integrate *either* over $p_{1,4}$ *or* over $p_{2,4}$, which just eliminates the delta function for energy conservation [more symmetrically, one may integrate over $(p_{1,4}+p_{2,4})$ instead]. Performing the integral over $q_4$ then converts Eq. (3.65) to Eq. (3.45).

Although it is certainly not surprising that the Lagrangian functional integral approach of Refs. [18,20,21] gives the same results for the equal-time correlation functions as our Hamiltonian approach, it is also not trivial. Concerning the gauge fixing procedure, there is the following important difference between the two approaches: in the Lagrangian formulation, the Weyl gauge $A_0 \equiv 0$ cannot be implemented in addition to the Coulomb gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A} \equiv 0$ [30]. Indeed, in the first-order Lagrangian formalism, integrating out

the $A_0$-field rather than setting $A_0$ to zero yields an expression in the exponent of the measure for the functional integral that resembles the Christ–Lee Hamiltonian [18]. The derivation of the Hamilton operator (3.5) by Christ and Lee [22], on the other hand, relies on the existence of a gauge transformation that makes any gauge field $A_\mu$ satisfy both the Coulomb and Weyl gauge conditions. We discuss the possibility of simultaneously implementing the Weyl and Coulomb gauges in the Hamiltonian and the Lagrangian approaches in the Appendix.

Although not defined as an equal-time correlation function in our approach, it turns out that the static potential (3.58)–(3.59) is related to the space-time two-point function $\langle A_0^a(p_1)A_0^b(p_2)\rangle$ in the Lagrangian functional integral approach, as was first pointed out by Zwanziger [31]. The formal expression for the space-time correlation function is [20]

$$\langle A_0^a(p_1)A_0^b(p_2)\rangle = \left(\frac{1}{\mathbf{p}_1^2} + \frac{N_c g^2}{\left(\mathbf{p}_1^2\right)^2}\, 3 \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{p_{1,i} p_{1,j}\, t_{ij}(\mathbf{q})}{q^2\,(\mathbf{p}_1-\mathbf{q})^2} \right. \tag{3.66}$$

$$\left. + \frac{N_c g^2}{\left(\mathbf{p}_1^2\right)^2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{q_4}{p_{1,4}} \frac{\mathbf{p}_1\cdot(\mathbf{p}_1-2\mathbf{q})}{q^2\,(p_1-q)^2}\, t_{ij}(\mathbf{p}_1-\mathbf{q}) t_{ij}(\mathbf{q}) \right) \delta^{ab}\,(2\pi)^4 \delta(p_1+p_2). \tag{3.67}$$

Integrating over either $p_{1,4}$ or $p_{2,4}$ and over the energy component $q_4$ of the loop momentum, we obtain the antiscreening contribution (3.58) to the static potential from Eq. (3.66) because the latter is already instantaneous. In order to find Eq. (3.59) starting from Eq. (3.67), we have to put the respective other energy component, $p_{2,4}$ or $p_{1,4}$, to zero in addition [this is not necessary in the cases of Eqs. (3.65) and (3.66), because there the result of integrating over one of the energy components is independent of the other]. For $\langle A_0^a(\mathbf{p}_1,t_1)A_0^b(\mathbf{p}_2,t_2)\rangle$, this procedure amounts to integrating over the relative time $t_1-t_2$, which is, in fact, intuitively quite appealing for a non-instantaneous contribution to the potential between static sources.

We shall now use the representation (3.61) of the equal-time gluon two-point function and the corresponding representations of the ghost two-point function and the static potential for the renormalization of these equal-time correlation functions. To this end, we make use of the explicit expressions obtained for Eqs. (3.62)–(3.67) in Ref. [20] in dimensional regularization. Thus, by integrating the result for (3.62)–(3.64) according to Eq. (3.61), we obtain for the equal-time correlation function

$$\langle A_i^a(\mathbf{p}_1)A_j^b(\mathbf{p}_2)\rangle = \left[\frac{1}{2|\mathbf{p}_1|} + \frac{N_c g^2}{(4\pi)^2}\frac{1}{2|\mathbf{p}_1|}\left(\frac{1}{\varepsilon} - \ln\frac{\mathbf{p}_1^2}{\mu^2} + C_A\right)\right]\delta^{ab}\, t_{ij}(\mathbf{p}_1)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2) \tag{3.68}$$

in the limit $\varepsilon\to 0$, where $d=3-2\varepsilon$ is the dimension of space and $\mu$ an arbitrary mass scale. The value of the constant $C_A$ is not relevant to our purposes, but being an integral over an explicitly known function of $p_{1,4}^2/\mathbf{p}_1^2$, we have carefully checked that it is finite.

From the explicit expressions for Eqs. (3.65)–(3.67) in dimensional regularization [20], we find directly

$$\langle c^a(\mathbf{p}_1)\bar{c}^b(\mathbf{p}_2)\rangle = \left[\frac{1}{\mathbf{p}_1^2} + \frac{N_c g^2}{(4\pi)^2}\frac{1}{\mathbf{p}_1^2}\frac{4}{3}\left(\frac{1}{\varepsilon} - \ln\frac{\mathbf{p}_1^2}{\mu^2} + C_c\right)\right]\delta^{ab}\,(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2), \tag{3.69}$$

$$V(\mathbf{p}_1) = \frac{1}{\mathbf{p}_1^2} + \frac{N_c g^2}{(4\pi)^2}\frac{1}{\mathbf{p}_1^2}\frac{11}{3}\left(\frac{1}{\varepsilon} - \ln\frac{\mathbf{p}_1^2}{\mu^2} + C_V\right). \tag{3.70}$$

This procedure to regularize the equal-time correlation functions finds further support in the cases where the equal-time functions in the form (3.42)–(3.45) and (3.58)–(3.59) can be evaluated directly in dimensional regularization (in $d = 3 - 2\varepsilon$ dimensions). For Eqs. (3.44) and (3.45), identical results are obtained in both ways [29] [also trivially for Eq. (3.42) and the loop integral in Eq. (3.58) which is just three times the one of Eq. (3.45)].

The results (3.68) and (3.69) for the equal-time two-point correlation functions can be renormalized in analogy to the procedures developed for covariant theories: we introduce renormalized correlation functions (or correlation functions of the renormalized fields)

$$\begin{aligned}\langle A^a_{R,i}(\mathbf{p}_1)A^b_{R,j}(\mathbf{p}_2)\rangle &= \frac{1}{Z_A}\langle A^a_i(\mathbf{p}_1)A^b_j(\mathbf{p}_2)\rangle,\\ \langle c^a_R(\mathbf{p}_1)\bar{c}^b_R(\mathbf{p}_2)\rangle &= \frac{1}{Z_c}\langle c^a(\mathbf{p}_1)\bar{c}^b(\mathbf{p}_2)\rangle.\end{aligned} \tag{3.71}$$

The simplest choice of the normalization conditions is

$$\begin{aligned}\langle A^a_{R,i}(\mathbf{p}_1)A^b_{R,j}(\mathbf{p}_2)\rangle\Big|_{\mathbf{p}_1^2=\kappa^2} &= \frac{1}{2|\mathbf{p}_1|}\,\delta^{ab}\,t_{ij}(\mathbf{p}_1)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2),\\ \langle c^a_R(\mathbf{p}_1)\bar{c}^b_R(\mathbf{p}_2)\rangle\Big|_{\mathbf{p}_1^2=\kappa^2} &= \frac{1}{\mathbf{p}_1^2}\,\delta^{ab}\,(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2),\end{aligned} \tag{3.72}$$

at the renormalization scale $\kappa$. With these normalization conditions and the results (3.68), (3.69), we obtain

$$\begin{aligned}Z_A(\kappa) &= 1 + \frac{N_c g^2}{(4\pi)^2}\left(\frac{1}{\varepsilon} - \ln\frac{\kappa^2}{\mu^2} + C_A\right):,\\ Z_c(\kappa) &= 1 + \frac{N_c g^2}{(4\pi)^2}\,\frac{4}{3}\left(\frac{1}{\varepsilon} - \ln\frac{\kappa^2}{\mu^2} + C_c\right),\end{aligned} \tag{3.73}$$

to order $g^2$.

The expression (3.70) for the static potential needs to be renormalized, too. This is most naturally achieved by a renormalization of the coupling constant as was first suggested in Refs. [32,33], through

$$g^2 V(\mathbf{p})\big|_{\mathbf{p}^2=\kappa^2} = \frac{\bar{g}^2_R(\kappa)}{\mathbf{p}^2}\,, \tag{3.74}$$

see Eq. (3.57). Hence, from Eq. (3.70),

$$\bar{g}^2_R(\kappa) = g^2\left[1 + \frac{N_c g^2}{(4\pi)^2}\,\frac{11}{3}\left(\frac{1}{\varepsilon} - \ln\frac{\kappa^2}{\mu^2} + C_V\right)\right], \tag{3.75}$$

which implies for the corresponding beta function to one-loop order,

$$\kappa^2\frac{\partial}{\partial\kappa^2}\,\bar{g}^2_R(\kappa) = \frac{\bar{\beta}_0}{(4\pi)^2}\,\bar{g}^4_R(\kappa), \tag{3.76}$$

that

$$\bar{\beta}_0 = -\frac{11}{3}\,N_c\,. \tag{3.77}$$

This is the well-known result from covariant perturbation theory (for Yang–Mills theory in covariant gauges), and has also been found in Ref. [20].

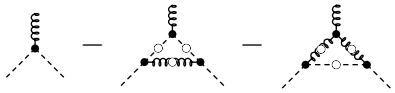

Figure 3.7: The proper ghost-gluon vertex to one-loop order.

For the rest of this section, we will pursue a more conventional way of renormalizing the coupling constant (which, however, leads to the same result). To this end, we consider the equal-time ghost-gluon three-point correlation function $\langle c^a(\mathbf{p}_1)\bar{c}^b(\mathbf{p}_2)A_i^c(\mathbf{p}_3)\rangle$ to order $g^3$ (one loop). The calculation of this correlation function is performed in analogy with the determination of the equal-time two-point correlation functions in Section 3.3, using the result for the vacuum wave functional obtained in Section 3.2. In this particularly simple case (and to the order considered), the external "propagators" (equal-time two-point functions) can be factorized to define the equal-time proper three-point vertex $\Gamma_i^{abc}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ as

$$\begin{aligned}\langle c^a(\mathbf{p}_1)\bar{c}^b(\mathbf{p}_2)A_i^c(\mathbf{p}_3)\rangle = -\int \frac{\mathrm{d}^3p_4}{(2\pi)^3}\frac{\mathrm{d}^3p_5}{(2\pi)^3}\frac{\mathrm{d}^3p_6}{(2\pi)^3}\,\langle c^a(\mathbf{p}_1)\bar{c}^d(-\mathbf{p}_4)\rangle \\ \times\,\Gamma_j^{def}(\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6)\,\langle c^e(-\mathbf{p}_5)\bar{c}^b(\mathbf{p}_2)\rangle\,\langle A_j^f(-\mathbf{p}_6)A_i^c(\mathbf{p}_3)\rangle. \quad (3.78)\end{aligned}$$

The explicit perturbative result is

$$\begin{aligned}\Gamma_j^{abc}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = -\mathrm{i}gf^{abc}t_{jk}(\mathbf{p}_3)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\Bigg(p_{1,k} \\ -\frac{N_\mathrm{c}g^2}{2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\big[\mathbf{p}_1\cdot\mathbf{p}_2-(\mathbf{p}_1\cdot\hat{\mathbf{q}})(\mathbf{p}_2\cdot\hat{\mathbf{q}})\big](p_{1,k}-q_k)}{2|\mathbf{q}|\,(\mathbf{p}_1-\mathbf{q})^2\,(\mathbf{p}_2+\mathbf{q})^2} \quad (3.79) \\ +\frac{2N_\mathrm{c}g^2}{2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{p_{1,l}p_{2,n}\,t_{lm}(\mathbf{p}_1-\mathbf{q})\,t_{nr}(\mathbf{p}_2+\mathbf{q})}{\mathbf{q}^2\,2|\mathbf{p}_1-\mathbf{q}|\,2|\mathbf{p}_2+\mathbf{q}|} \\ \times\frac{\delta_{km}(p_{1,r}-p_{3,r}-q_r)-\delta_{mr}(p_{1,k}-p_{2,k}-2q_k)-\delta_{rk}(p_{2,m}-p_{3,m}+q_m)}{|\mathbf{q}|+|\mathbf{p}_1-\mathbf{q}|+|\mathbf{p}_2+\mathbf{q}|}\Bigg) \quad (3.80)\end{aligned}$$

It is represented diagrammatically in Fig. 3.7. Note that due to the transversality of the gauge, two powers of the external momenta can be factorized from the loop integrals [cf. Eq. (3.45)] and, as a result, the integrals are UV finite. This phenomenon is well-known in another transverse gauge, the Landau gauge [34,35]. For future use, we note that by very lengthy algebra the tensor structure in Eq. (3.80) can be simplified as follows:

$$\begin{aligned}p_{1,l}\,p_{2,n}\,t_{lm}(\mathbf{p}_1-\mathbf{q})t_{nr}(\mathbf{p}_2+\mathbf{q})\Big[\delta_{km}(p_{1,r}-p_{3,r}-q_r) \\ -\delta_{mr}(p_{1,k}-p_{2,k}-2q_k)-\delta_{rk}(p_{2,m}-p_{3,m}+q_m)\Big]t_{jk}(\mathbf{p}_3)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) = \\ = 2\Bigg(\mathbf{q}^2p_{1,k}+(\mathbf{p}_1\cdot\mathbf{p}_2)q_k-\frac{(\mathbf{p}_1-\mathbf{q})\cdot(\mathbf{p}_2+\mathbf{q})}{(\mathbf{p}_1-\mathbf{q})^2\,(\mathbf{p}_2+\mathbf{q})^2}\Big\{[\mathbf{q}\cdot(\mathbf{p}_1-\mathbf{q})][\mathbf{q}\cdot(\mathbf{p}_2+\mathbf{q})]p_{1,k} \\ +[\mathbf{p}_1\cdot(\mathbf{p}_1-\mathbf{q})][\mathbf{p}_2\cdot(\mathbf{p}_2+\mathbf{q})]q_k\Big\}\Bigg)\,t_{jk}(\mathbf{p}_3)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3). \quad (3.81)\end{aligned}$$

We define the renormalized coupling constant in analogy with the covariant case as

$$\Gamma^{abc}_{R,j}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\Big|_{\mathbf{p}_1^2=\mathbf{p}_2^2=\mathbf{p}_3^2=\kappa^2} \equiv Z_c(\kappa)Z_A^{1/2}(\kappa)\,\Gamma^{abc}_j(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)\Big|_{\mathbf{p}_1^2=\mathbf{p}_2^2=\mathbf{p}_3^2=\kappa^2}$$
$$= -\mathrm{i}g_R(\kappa)f^{abc}\,p_{1,k}\,t_{jk}(\mathbf{p}_3)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \qquad (3.82)$$

at the symmetric point. As a consequence, using Eq. (3.73) and the UV finiteness of the loop integrals (3.79)–(3.80),

$$g_R(\kappa) = g\left[1+\frac{N_\mathrm{c}g^2}{(4\pi)^2}\,\frac{11}{6}\left(\frac{1}{\varepsilon}-\ln\frac{\kappa^2}{\mu^2}+C\right)\right], \qquad (3.83)$$

with a finite constant $C$ given by $(11/6)C = (4/3)C_c + (1/2)C_A + C_v$, where $C_v$ is obtained from the finite loop integrals in Eqs. (3.79)–(3.80).

For the beta function defined in analogy with Eq. (3.76) we obtain from Eq. (3.83)

$$\beta_0 = -\frac{11}{3}\,N_\mathrm{c}\,, \qquad (3.84)$$

which coincides with the one obtained in Eq. (3.77) before with the renormalized coupling constant defined through the static potential. We should mention that we could have extracted the beta function directly from the cut-off dependence of the vacuum wave functional, as it has actually been done in Ref. [26] for Yang–Mills theory in Weyl gauge. Here, however, our intention was to closely follow the procedure applied in the Lagrangian covariant formulation.

The integration of the renormalization group equation (3.76) gives the well-known (one-loop) result

$$g_R^2(\kappa) = \frac{(4\pi)^2}{\dfrac{11}{3}N_\mathrm{c}\ln\left(\dfrac{\kappa^2}{\Lambda^2_\mathrm{QCD}}\right)} \qquad (3.85)$$

[and the same for $\bar{g}_R^2(\kappa)$ (3.75)]. It must be noted that for renormalization group improvements like Eq. (3.85) to be sensible we have to suppose that the three-dimensional formulation presented here is multiplicatively renormalizable to all orders in the same way as the usual formulation of a renormalizable covariant quantum field theory, which is not known at present (even the renormalizability of the Lagrangian functional integral approach to Coulomb gauge Yang–Mills theory has not yet been shown). Equation (3.85) and the developments to follow are therefore to some degree speculative, but it seemed of some interest to us to explore the consequences of the natural assumption of multiplicative renormalizability.

With these qualifications, we go on to use a standard renormalization group argument to extract the asymptotic UV behavior of the equal-time two-point correlation functions. To this end, we differentiate Eq. (3.71) with respect to $\kappa^2$ using the $\kappa$-independence of the "bare" two-point functions. It is then seen that the $\kappa$-dependence of the renormalized two-point functions is determined by the anomalous dimensions $(\kappa^2\,\partial \ln Z_{A,c}/\partial\kappa^2)$. Evaluating the latter from Eq. (3.73) and replacing $g^2$ in the results with $g_R^2(\kappa)$, we obtain the desired renormalization group equations for the equal-time two-point functions, explicitly

$$\kappa^2\frac{\partial}{\partial\kappa^2}\,\langle A^a_{R,i}(\mathbf{p}_1)A^b_{R,j}(\mathbf{p}_2)\rangle = \frac{N_\mathrm{c}g_R^2(\kappa)}{(4\pi)^2}\,\langle A^a_{R,i}(\mathbf{p}_1)A^b_{R,j}(\mathbf{p}_2)\rangle,$$
$$\kappa^2\frac{\partial}{\partial\kappa^2}\langle c^a_R(\mathbf{p}_1)\bar{c}^b_R(\mathbf{p}_2)\rangle = \frac{4}{3}\,\frac{N_\mathrm{c}g_R^2(\kappa)}{(4\pi)^2}\,\langle c^a_R(\mathbf{p}_1)\bar{c}^b_R(\mathbf{p}_2)\rangle. \qquad (3.86)$$

In these equations, we substitute from Eq. (3.85) for $g_R^2(\kappa)$ and integrate. Using the normalization conditions (3.72) for the determination of the integration constants, one obtains the momentum dependence of the equal-time two-point functions:

$$\begin{aligned}\langle A^a_{R,i}(\mathbf{p}_1)A^b_{R,j}(\mathbf{p}_2)\rangle &= \frac{1}{2|\mathbf{p}_1|}\left(\frac{\ln\left(\dfrac{\kappa^2}{\Lambda^2_{\mathrm{QCD}}}\right)}{\ln\left(\dfrac{\mathbf{p}_1^2}{\Lambda^2_{\mathrm{QCD}}}\right)}\right)^{3/11}\delta^{ab}\,t_{ij}(\mathbf{p}_1)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2),\\ \langle c^a_R(\mathbf{p}_1)\bar{c}^b_R(\mathbf{p}_2)\rangle &= \frac{1}{\mathbf{p}_1^2}\left(\frac{\ln\left(\dfrac{\kappa^2}{\Lambda^2_{\mathrm{QCD}}}\right)}{\ln\left(\dfrac{\mathbf{p}_1^2}{\Lambda^2_{\mathrm{QCD}}}\right)}\right)^{4/11}\delta^{ab}\,(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2).\end{aligned} \tag{3.87}$$

The momentum dependence of the "bare" two-point functions, obtained from Eq. (3.87) simply by multiplying with the corresponding wave function renormalization constants $Z_{A,c}$, is obviously the same. By solving the renormalization group equations for $Z_A$ and $Z_c$ that involve the anomalous dimensions, it may be shown explicitly that the bare two-point functions are $\kappa$-independent, as they must be.

For the static potential, on the other hand, we immediately obtain from Eqs. (3.74) and (3.85) [for $\bar{g}_R^2(\kappa)$] the renormalization group improved result

$$g^2V(\mathbf{p}) = \frac{(4\pi)^2}{\dfrac{11}{3}N_{\mathrm{c}}\,\mathbf{p}^2\ln\left(\dfrac{\mathbf{p}^2}{\Lambda^2_{\mathrm{QCD}}}\right)}\,. \tag{3.88}$$

Note that this one-loop formula constitutes a very direct expression of asymptotic freedom.

The result (3.87) for the momentum dependence of the equal-time two-point functions has also been obtained in Ref. [36] from a Dyson–Schwinger equation for the equal-time ghost correlator, where the gauge-invariant one-loop running (3.85) of the renormalized coupling constant is used as an input. We briefly discuss that derivation here, adapted to the conventions of the present paper.

The renormalized equal-time two-point functions are parameterized as

$$\langle A^a_{R,i}(\mathbf{p}_1)A^b_{R,j}(\mathbf{p}_2)\rangle = \frac{1}{2\omega(\mathbf{p}_1^2)}\,\delta^{ab}\,t_{ij}(\mathbf{p}_1)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2) \tag{3.89}$$

and

$$\langle c^a_R(\mathbf{p}_1)\bar{c}^b_R(\mathbf{p}_2)\rangle = \frac{d(\mathbf{p}_1^2)}{\mathbf{p}_1^2}\,\delta^{ab}\,(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2), \tag{3.90}$$

and normalized according to the conditions (3.72). The renormalized coupling constant is defined as before in Eq. (3.82). Then the Dyson–Schwinger equation for the equal-time ghost two-point function reads [3–6]

$$d^{-1}(\mathbf{p}^2) = Z_c - N_{\mathrm{c}}\,g_R^2(\kappa)\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2}{2\omega(\mathbf{q}^2)}\,\frac{d\big((\mathbf{p}-\mathbf{q})^2\big)}{(\mathbf{p}-\mathbf{q})^2}. \tag{3.91}$$

Here we have approximated the full ghost-gluon vertex appearing in the exact equation by the tree-level vertex, as it is appropriate in order to obtain the (renormalization-group improved) one-loop expressions.

In order to solve Eq. (3.91), we make the following, properly normalized, ansatzes for the two-point functions in the ultraviolet limit $\mathbf{p}^2 \gg \Lambda_{\mathrm{QCD}}^2$,

$$\frac{|\mathbf{p}|}{\omega(\mathbf{p}^2)} = \left( \frac{\ln\left(\dfrac{\kappa^2}{\Lambda_{\mathrm{QCD}}^2}\right)}{\ln\left(\dfrac{\mathbf{p}^2}{\Lambda_{\mathrm{QCD}}^2}\right)} \right)^{\gamma}, \qquad d(\mathbf{p}^2) = \left( \frac{\ln\left(\dfrac{\kappa^2}{\Lambda_{\mathrm{QCD}}^2}\right)}{\ln\left(\dfrac{\mathbf{p}^2}{\Lambda_{\mathrm{QCD}}^2}\right)} \right)^{\delta}, \tag{3.92}$$

with the exponents $\gamma$ and $\delta$ to be determined. The integral in Eq. (3.91) can then be calculated in the limit $\mathbf{p}^2 \gg \Lambda_{\mathrm{QCD}}^2$ and the Dyson–Schwinger equation yields the relation [36]

$$\ln^{-\delta}\left(\frac{\kappa^2}{\Lambda_{\mathrm{QCD}}^2}\right) \ln^{\delta}\left(\frac{\mathbf{p}^2}{\Lambda_{\mathrm{QCD}}^2}\right) = N_{\mathrm{c}}\, g_R^2(\kappa) \frac{1}{(4\pi)^2} \frac{4}{3\delta} \ln^{\gamma+\delta}\left(\frac{\kappa^2}{\Lambda_{\mathrm{QCD}}^2}\right) \ln^{1-\gamma-\delta}\left(\frac{\mathbf{p}^2}{\Lambda_{\mathrm{QCD}}^2}\right), \tag{3.93}$$

from which we infer the sum rule

$$\gamma + 2\delta = 1 \tag{3.94}$$

for the exponents as well as the identity

$$g_R^2(\kappa) \frac{1}{(4\pi)^2} \frac{4}{3\delta} N_{\mathrm{c}} \ln\left(\frac{\kappa^2}{\Lambda_{\mathrm{QCD}}^2}\right) = 1 \tag{3.95}$$

for the coefficients. Consistency of the latter relation with the well-known perturbative result (3.85) yields the exponents

$$\gamma = \frac{3}{11}, \qquad \delta = \frac{4}{11}, \tag{3.96}$$

where we have used the sum rule (3.94) again. We have thus regained the result of Eq. (3.87).

## 3.5 Conclusions

In this work, we have accomplished a systematic perturbative solution of the Yang–Mills Schrödinger equation in Coulomb gauge for the vacuum wave functional following the $\mathrm{e}^S$ method in many-body physics. This resulted in a functional integral representation for the calculation of equal-time correlation functions. We have derived a diagrammatical representation of these functions, order by order in perturbation theory, where the vertices in the diagrams are determined from the perturbative calculation of the vacuum wave functional. The number of the vertices, which by themselves have a perturbative expansion, grows with the perturbative order. We have determined the equal-time gluon and ghost two-point correlation functions and the potential between static colour charges to one-loop order in this way.

The results coincide with those of a straightforward calculation in Rayleigh–Schrödinger perturbation theory [29], and also with the values for equal times of the two-point space-time correlation functions from a Lagrangian functional integral representation [20]. We

have emphasized that the latter coincidence is not trivial since the gauge fixing procedures in the Hamiltonian and the Lagrangian approach are profoundly different. We have also used the results of the Lagrangian approach to renormalize the equal-time two-point correlation functions and the static potential.

With the help of the non-renormalization of the ghost-gluon vertex which we also show, or, alternatively, from the static potential, we have extracted the running of the correspondingly defined renormalized coupling constant. The result for the beta function is the one also found in covariant and other gauges, $\beta_0 = -(11/3)N_c$ to one-loop order. We have used standard renormalization group arguments to determine the asymptotic ultraviolet behavior of the equal-time two-point functions and the static potential under the assumption of multiplicative renormalizability to all orders, with the result that

$$\begin{aligned}
\langle A_i^a(\mathbf{p}_1) A_j^b(\mathbf{p}_2)\rangle &\propto \frac{\big(\ln(\mathbf{p}_1^2/\Lambda_{\mathrm{QCD}}^2)\big)^{-3/11}}{2|\mathbf{p}_1|}\,\delta^{ab}\,t_{ij}(\mathbf{p}_1)(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2),\\
\langle c^a(\mathbf{p}_1)\bar{c}^b(\mathbf{p}_2)\rangle &\propto \frac{\big(\ln(\mathbf{p}_1^2/\Lambda_{\mathrm{QCD}}^2)\big)^{-4/11}}{\mathbf{p}_1^2}\,\delta^{ab}\,(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2),\\
g^2 V(\mathbf{p}_1) &\propto \frac{\big(\ln(\mathbf{p}_1^2/\Lambda_{\mathrm{QCD}}^2)\big)^{-1}}{\mathbf{p}_1^2}
\end{aligned} \tag{3.97}$$

to one-loop order in the perturbative (asymptotically free) regime.

It is clear from the presence of an infinite number of vertices in the functional integral representation of the equal-time correlation functions (to infinite perturbative order) that the corresponding Dyson–Schwinger equations contain an infinite number of terms, a very serious problem for the determination of an appropriate approximation scheme for non-perturbative solutions. The existence of simplified diagrammatic rules for the calculation of equal-time correlation functions via the E-operator, to be appropriately extended to all perturbative orders, seems to point toward the possibility of formulating similar non-perturbative equations with a finite number of terms. It would indeed be very interesting to repeat the type of infrared analysis applied before to Yang–Mills theory in the Landau gauge [7,12–17] and to a variational ansatz in the Coulomb gauge [1–7] for such a set of equations.

The perturbative expression for the vacuum wave functional determined in this paper, in particular the coefficient functions $f_3$ and $f_4$ in Eqs. (3.22)–(3.25), can be used to motivate improved ansatzes for the vacuum functional going beyond the Gaussian form in a variational approach as employed in Refs. [1,3–6]. An appropriate extension of a Gaussian ansatz would make it possible, e.g. to reproduce the correct beta function and the anomalous dimensions of the equal-time two-point functions at least to one-loop order in this approach. This idea is currently being pursued. We note in this context that the relation (3.26) between the coefficient functions $f_2$ and $f_4$ following from the Schrödinger equation can easily be promoted to an exact gap equation, i.e. valid to any perturbative order, by replacing $|\mathbf{p}|$ and $|\mathbf{q}|$ on the right-hand side with $f_2(\mathbf{p})$ and $f_2(\mathbf{q})$, respectively:

$$\begin{aligned}
\big(f_2(\mathbf{p})\big)^2\delta^{ab}\delta_{ij} &= \left(\mathbf{p}^2 - f_2(\mathbf{p})\,\frac{N_c g^2}{2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2}{(\mathbf{p}-\mathbf{q})^2}\right)\delta^{ab}\delta_{ij}\\
&\quad + \frac{1}{2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,f_{4;ijkl}^{abcc}(-\mathbf{p},\mathbf{p},-\mathbf{q},\mathbf{q})t_{kl}(\mathbf{q}) - N_c g^2\delta^{ab}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{f_2(\mathbf{p})-f_2(\mathbf{q})}{(\mathbf{p}-\mathbf{q})^2}\,t_{ij}(\mathbf{q}).
\end{aligned} \tag{3.98}$$

Similarly, the explicit perturbative expression for the vacuum wave functional in the presence of static external colour charges, see Eqs. (3.49) and (3.55), is expected to provide valuable information in the quest for a detailed understanding of the (quenched) interaction between quarks. Different ansatzes for such a wave functional in the non-perturbative regime have been considered in Refs. [36,37], while relevant results from lattice calculations can be found in Refs. [38,39].

Finally, the results (3.97) for the ultraviolet behavior of the two-point functions are relevant to the corresponding results of a numerical evaluation on space-time lattices. A recent numerical calculation of the equal-time ghost two-point function [9] gives a value of 0.33(1) for the anomalous dimension, quite close to our one-loop result $4/11 \approx 0.36$. As for the static potential, numerical results are only available for the instantaneous antiscreening part, given to one-loop order by Eq. (3.58). The asymptotic behavior of this so-called colour Coulomb potential has been determined in Ref. [40] to be

$$g^2 V_{\mathrm{C}}(\mathbf{p}_1) \propto \frac{\left(\ln(\mathbf{p}_1^2/\Lambda_{\mathrm{QCD}}^2)\right)^{-1}}{\mathbf{p}_1^2} \tag{3.99}$$

to one-loop order, the same as for the full static potential (except for an overall factor 12/11). This ultraviolet behavior was confirmed by earlier lattice calculations [41–43]. However, the most recent numerical evaluation [44] contradicts these earlier findings.

The situation is even more controversial for the equal-time (transverse spatial) gluon two-point function. Some years ago, scaling violations had been reported by several groups [43,45,46]. Different proposals to understand or deal with the scaling violations have lead to different results in the most recent lattice simulations, from zero anomalous dimension [8,47] to an anomalous power behavior [9]

$$\langle A_i^a(\mathbf{p}_1) A_j^b(\mathbf{p}_2)\rangle \propto \frac{|\mathbf{p}_1|^{-\eta}}{2|\mathbf{p}_1|}\, \delta^{ab}\, t_{ij}(\mathbf{p}_1)(2\pi)^3 \delta(\mathbf{p}_1 + \mathbf{p}_2) \tag{3.100}$$

with $\eta = 0.40(2)$ (similar to results reported earlier [42]; see, however, the corresponding remarks in Ref. [46]). Clearly, a better understanding of the numerical data is needed. An interesting possibility in this context is the use of anisotropic lattices where a drastic reduction of the scaling violations was reported [48] in the approach to the Hamiltonian limit $a_s/a_t \to \infty$ (with the spatial and temporal lattice spacings $a_s$ and $a_t$).

Given that the question of multiplicative renormalizability of the gluonic two-point correlation function (at equal and at different times) plays an important role in the controversy about the scaling violations [8,9], it would certainly be interesting to confirm multiplicative renormalizability in our approach to the two-loop level. The extension of the calculations presented here to two loops appears relatively straightforward, albeit lengthy. Let us mention, in this context, that the static potential has never been worked out explicitly in Coulomb gauge at the two-loop level. In our approach, the main problem with the two-loop calculations is expected to be the correct renormalization of the expressions. Corresponding results for the space-time correlation functions from a Lagrangian functional integral approach are not known to this level, so one cannot proceed in analogy with Section 3.4. The calculations with the Lagrangian functional integral method are complicated, at the two-loop level, by the appearance of Christ–Lee–Schwinger terms [22] (see also Refs. [49,50]) which are required in order to produce results corresponding to the Hamiltonian (3.5) with its specific operator ordering (in particular, the insertion of

powers of the Faddeev–Popov determinant). As an alternative to the procedure of Section 3.4, one may consider the use of a simple three-dimensional ultraviolet momentum cut-off. Care has to be taken, however, since such a cut-off can break the covariance of the theory (see Ref. [51] for an example in Yukawa theory), in which case appropriate non-covariant counterterms have to be included.

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## Appendix: Gauge transformations in Lagrangian and Hamiltonian formalisms

While the Lagrangian approach to Yang–Mills theory offers some convenient features (such as manifestation of Lorentz invariance), the more cumbersome Hamiltonian approach yields equations of motion invariant under a larger set of gauge transformations. In what follows, we discuss gauge invariance starting from the classical Lagrangian and Hamiltonian functions, respectively, prior to quantization. We will employ standard covariant notation in this appendix; in particular, spatial subindices refer to the covariant components of the corresponding 4-vector or tensor.

The Lagrangian function of the gauge sector,

$$L = -\frac{1}{4}\int \mathrm{d}^3x\, F^a_{\mu\nu}(x)\, F^{\mu\nu}_a(x), \tag{3.101}$$

is invariant under gauge transformations of the gauge field $A_\mu(x) \equiv A^a_\mu(x)T^a$,

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{1}{g}\,U(x)\partial_\mu U^\dagger(x), \tag{3.102}$$

where $U \in SU(N)$ and $[T^a, T^b] = f^{abc}T^c$.

The Weyl gauge, $A^a_0(x) = 0$, can be found by choosing the time-ordered exponential

$$U^\dagger(x) = \mathsf{T}\exp\left(-g\int^t \mathrm{d}t' A_0(\mathbf{x}, t')\right). \tag{3.103}$$

To remain in the Weyl gauge, the transformation (3.103) may be followed by time-independent transformations $U(\mathbf{x})$ only. We can therefore fix the Coulomb gauge, $\partial_i A^i_a(x) = 0$, at one instant of time but it is impossible to fix both gauges simultaneously for all times.

In the Hamiltonian formalism, on the other hand, gauge transformations are generated by (first-class) constraints in configuration space [52]. To see that, supplement the Hamiltonian function

$$H = \frac{1}{2}\int \mathrm{d}^3x \left(\mathbf{\Pi}^2_a(x) + \mathbf{B}^2_a(x)\right) - \int \mathrm{d}^3x\, A^a_0(x)\hat{D}^{ab}_i(x)\Pi^i_b(x) \tag{3.104}$$

by the constraints

$$\phi_1^a(x) = \Pi_0^a(x) \approx 0, \qquad \phi_2^a(x) = \hat{D}_i^{ab}(x)\Pi_b^i(x) \approx 0 \tag{3.105}$$

with some arbitrary Lagrange multiplier fields $\{\lambda_k^a(x)\}$,

$$H_E = H + \sum_{k=1,2} \int \mathrm{d}^3x \; \lambda_k^a(x)\phi_k^a(x). \tag{3.106}$$

We defined $\Pi_\mu^a(x) = F_{\mu 0}^a(x)$ and $\hat{D}_i^{ab}(x) = \delta^{ab}\partial_i - g f^{abc} A_i^c(x)$. The extended Hamiltonian $H_E$ in Eq. (3.106) is equivalent to the original Hamiltonian $H$ since the constraints $\{\phi_k^a(x)\}$ vanish weakly (in the Dirac sense [52]). The infinitesimal time evolution of the gauge field $A_\mu^a(\mathbf{x}, t)$ from $t_0$ to $t = t_0 + \delta t$, generated by $H_E$ through the Poisson brackets,

$$A_\mu^a(\mathbf{x}, t) = A_\mu^a(\mathbf{x}, t_0) + \delta t \, \{A_\mu^a(\mathbf{x}, t_0), H\} + \delta t \sum_{k=1,2} \int \mathrm{d}^3y \; \lambda_k^b(y) \, \{A_\mu^a(\mathbf{x}, t_0), \phi_k^b(y)\}, \tag{3.107}$$

gives for two different sets of Lagrange multiplier functions $\{\lambda_k'^b(x)\}$ and $\{\lambda_k''^b(x)\}$ two different results $A_\mu'^a$ and $A_\mu''^a$, respectively. These differ to $\mathcal{O}(\delta t)$ by

$$A_\mu''^a(\mathbf{x}, t) - A_\mu'^a(\mathbf{x}, t) = \delta t \sum_{k=1,2} \int \mathrm{d}^3y \left(\lambda_k''^b(y) - \lambda_k'^b(y)\right) \{A_\mu^a(\mathbf{x}, t), \phi_k^b(y)\} \tag{3.108}$$

and are physically equivalent. Thus, the function

$$G = \sum_{k=1,2} \int \mathrm{d}^3y \; \tau_k^a(y)\phi_k^a(y) \tag{3.109}$$

generates infinitesimal gauge transformations in the (extended) Hamiltonian formalism with arbitrary functions $\tau_1^a(x)$ and $\tau_2^a(x)$. Computing the Poisson brackets in Eq. (3.108) yields

$$A_0^a(x) \to A_0^a(x) + \tau_1^a(x), \tag{3.110}$$

$$A_i^a(x) \to A_i^a(x) - \hat{D}_i^{ab}(x)\tau_2^b(x). \tag{3.111}$$

The difference to the gauge transformations (3.102) in the Lagrangian formalism is that the time component and the spatial components of the gauge field transform independently. The two functions $\tau_1^a(x)$ and $\tau_2^a(x)$ allow for a larger set of gauge transformations than the single function $U(x)$ in the Lagrangian formalism. The simultaneous fixing of Weyl and Coulomb gauges, which is impossible in the Lagrangian formalism, can be accomplished in the Hamiltonian formalism by appropriately choosing $\tau_1^a(x)$ and $\tau_2^a(x)$ (see Ref. [53] for the abelian case). Subsequently, the non-abelian gauge-fixed theory can be canonically quantized with projection on the physical Hilbert space [22], or with Dirac brackets [36] enforcing all constraints strongly. Both quantization prescriptions produce the Hamiltonian operator given by Eq. (3.5).

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Aliena vitia in oculis habemus,
a tergo nostra sunt.

*(Lucius Annaeus Seneca)*

# Article 4

# Non-Gaussian wave functionals in Coulomb gauge Yang–Mills theory

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**Abstract** A general method to treat non-Gaussian vacuum wave functionals in the Hamiltonian formulation of a quantum field theory is presented. By means of Dyson–Schwinger techniques, the static Green functions are expressed in terms of the kernels arising in the Taylor expansion of the exponent of the vacuum wave functional. These kernels are then determined by minimizing the vacuum expectation value of the Hamiltonian. The method is applied to Yang–Mills theory in Coulomb gauge, using a vacuum wave functional whose exponent contains up to quartic terms in the gauge field. An estimate of the cubic and quartic interaction kernels is given using as input the gluon and ghost propagators found with a Gaussian wave functional.

## 4.1 Introduction

According to our present understanding of nature, Quantum Chromodynamics (QCD) is the theory of the strong interaction. This theory has been tested in the high-momentum or ultraviolet (UV) regime, where perturbation theory is applicable due to asymptotic freedom. Our knowledge on the low energy, strongly interacting regime of QCD stems mainly from lattice calculations, which have at least qualitatively reproduced many physical observables, in particular, the linearly rising confining potential for heavy quarks. Furthermore, lattice calculations have revealed the relevance of topological field configurations such as magnetic monopoles and center vortices for infrared phenomena like confinement and spontaneous chiral symmetry breaking. These calculations support the dual Meissner effect and the vortex condensation picture of confinement [1]. Despite these substantial physical insights provided by the lattice calculations, a thorough understanding of these infrared phenomena will not come from lattice calculations alone but will require also studies of the continuum theory.

In recent years there have been substantial efforts devoted to a non-perturbative treatment of continuum Yang–Mills theory. Among these are a variational solution of the Yang–Mills Schrödinger equation in Coulomb gauge [2–5]. In this approach, using Gaussian type wave functionals, minimization of the energy density results in the so-called gap equation for the gluon energy (or static gluon propagator). This equation has been solved analytically in the infrared [6] and in the ultraviolet [7] and numerically in the full momentum regime [4,8]. One finds a gluon energy, which in the UV behaves like the photon energy but diverges in the infrared (IR), signalling confinement. The obtained gluon energy also compares favourably with the lattice data [9]. In particular, the infrared regime is correctly reproduced, as far as we can tell from available lattice data. There are, however, deviations in the mid-momentum regime (and minor ones in the UV) which can be attributed to the missing gluon loop, which escapes the Gaussian wave functionals. These deviations are presumably irrelevant for the confinement properties, which are dominated by the ghost loop (which is fully included under the Gaussian ansatz), but are believed to be important for a correct description of spontaneous breaking of chiral symmetry [10].

The numerical wave functional obtained from the variational solution of Ref. [8] seems to embody the correct infrared physics as is revealed in the various applications considered to date: one finds a linearly rising static quark potential [8], an infrared enhanced running coupling constant with no Landau pole [6], a topological susceptibility in accord with lattice data [11], a perimeter law for the 't Hooft loop [12], and, within an approximate Dyson–Schwinger equation, an area law for the spatial Wilson loop [13]. Furthermore, in Ref. [14] it was shown that the inverse ghost form factor of Coulomb gauge Yang–Mills theory represents the dielectric function of the Yang–Mills vacuum and the so-called horizon condition [15] (of an infrared diverging ghost form factor) implies that the Yang–Mills vacuum is a perfect color dielectricum, i.e. a dual superconductor, which establishes the connection between the Gribov–Zwanziger confinement scenario [15,16] and the monopole condensation picture [17,18]. Finally, in Ref. [19] the functional renormalization group flow equation of the Hamiltonian approach to Coulomb gauge Yang–Mills theory was studied, yielding results for the gluon and ghost propagator similar to that of the variational approach [4].

In the present paper, we generalize the variational approach to the Hamiltonian formulation of Yang–Mills theory to non-Gaussian wave functionals. We will present a general method to treat non-Gaussian wave functionals in quantum field theory. The method is based on the observation that expectation values in the Hamiltonian formulation of $d = 3 + 1$ dimensional quantum field theory can be formally obtained from a generating functional of $d = 3$ dimensional Euclidean quantum field theory with an action defined by the logarithm of the vacuum wave functional. Expanding this action functional in powers of the underlying field results in "bare" $n$-point kernels $\gamma_n$ as expansion "coefficients". We then exploit Dyson–Schwinger equation (DSE) techniques [20–22] to express the expectation value of the Hamiltonian $\langle H \rangle$ in terms of these kernels $\gamma_n$, which are then determined by the variational principle, i.e. by minimizing $\langle H \rangle$. This approach is then applied to the Hamiltonian formulation of Yang–Mills theory in Coulomb gauge to include three- and four-gluon interaction kernels in the exponent of the Yang–Mills vacuum wave functional.

By using such a non-Gaussian wave functional, the gluon loop is retained in the expectation value of the Hamiltonian. Although the gluon loop is irrelevant for the IR properties, it certainly influences the mid-momentum and UV regime of the gluon propagator and thus of the running coupling, and also contributes to the anomalous dimensions. As a first

estimate of the effects of the non-Gaussian terms in the wave functional, we will calculate the gluon-loop contribution to the gluon propagator as well as the three- and four-gluon proper vertices using the ghost and gluon propagators obtained from the Gaussian wave functional [8] as input. A full self-consistent inclusion of the three- and four-gluon vertices will be the subject of future research.

It is clear from the very beginning that eventually we have to truncate the tower of DSEs for the proper $n$-point vertex functions $\Gamma_n$ as well as the equations of motion for the variational kernels $\gamma_n$ following from the variational principle. For a systematic counting of the various diagrams we will assume a skeleton expansion.

Previous variational calculations (using Gaussian wave functionals) were restricted to two (overlapping) loops in the energy $\langle H \rangle$, resulting in a one-loop gap equation for the gluon propagator. Restriction to two overlapping loops in the energy results in a bare (zero-loop) three-gluon kernel $\gamma_3$ and in a vanishing four-gluon kernel, $\gamma_4 = 0$. To get a $\gamma_4 \neq 0$, one has to include up to three loops in the energy. To keep the calculation sufficiently simple, we will keep only those three (overlapping) loop terms in the energy containing three- or four-gluon kernels. This will result in a bare (zero-loop) four-gluon and a one-loop three-gluon vertex.

The organization of the paper is as follows: in Sec. 4.2 we present the DSEs of the Hamiltonian approach first for a general field theory and afterwards for Yang–Mills theory in Coulomb gauge. The full static (equal-time) propagators of the Hamiltonian approach are expressed in terms of proper vertex functions in Sec. 4.3. In Sec. 4.4 we specify our Yang–Mills vacuum wave functional and derive the corresponding DSEs for the gluon and ghost proper $n$-point functions. By means of these DSEs, the vacuum expectation value of the Hamiltonian is expressed in Sec. 4.5 in terms of the variational kernels of the vacuum wave functional. In Sec. 4.6 these kernels are determined by minimizing the energy density. Finally, in Sec. 4.7 we calculate the three- and four-gluon proper vertices using as input the ghost and gluon propagators from the variational calculations with a Gaussian wave functional. A short summary and our conclusions are given in Sec. 4.8.

## 4.2 Dyson–Schwinger equations of the Hamiltonian approach to Yang–Mills theory in Coulomb gauge

### 4.2.1 General DSE formalism of the Hamiltonian approach to quantum field theory

Consider a quantum field theory comprised of a collection of fields $\phi = (\phi_1, \phi_2, \dots)$ and let $|\psi\rangle$ be the exact vacuum state. All static (time-independent) Green's functions, i.e. vacuum expectation values $\langle \phi\phi \dots \rangle$, can be calculated from the generating functional

$$Z[j] = \langle \psi | \, \mathrm{e}^{\int j \cdot \phi} \, | \psi \rangle, \tag{4.1}$$

where $j = (j_1, j_2, \dots)$ stands for the collection of sources corresponding to the fields and we use the abbreviation $j \cdot \phi = j_1\phi_1 + j_2\phi_2 + \cdots$. In the "coordinate" representation of the vacuum state $\langle \phi | \psi \rangle = \psi[\phi]$, the scalar product in Eq. (4.1) is defined by the functional integral over time-independent fields $\phi(\mathbf{x})$

$$Z[j] = \int \mathcal{D}\phi \, |\psi[\phi]|^2 \, \mathrm{e}^{\int j \cdot \phi}. \tag{4.2}$$

Furthermore, the integral in the exponent is over spatial coordinates $\mathbf{x}$ of the static fields $\phi(\mathbf{x})$. Expressing the vacuum wave functional in the form[1]

$$\psi[\phi] = \exp\left(-\frac{1}{2}\,S[\phi]\right), \tag{4.3}$$

the generating functional of the Hamiltonian approach to quantum field theory becomes

$$Z[j] = \int \mathcal{D}\phi\, \mathrm{e}^{-S[\phi]+\int j\cdot\phi}, \tag{4.4}$$

which is a standard generating functional of the $d = 3$-dimensional Euclidean quantum field theory defined by an "action" $S[\phi]$. Here, this action is defined by the vacuum wave functional $\psi[\phi]$ and will, in general, be non-local and non-linear. We therefore perform a Taylor expansion of the action functional $S[\phi]$ in powers of the time-independent fields $\phi(\mathbf{x})$. The constant part $S[0]$ is fixed by the normalization of the wave functional and the linear part can be absorbed into the external source. It is then sufficient to consider expansions of $S[\phi]$ starting at second order

$$S[\phi] = \frac{1}{2}\int \gamma_2\,\phi^2 + \frac{1}{3!}\int \gamma_3\,\phi^3 + \cdots\,. \tag{4.5}$$

Restricting the expansion to second order yields a Gaussian wave functional for which the functional integral in Eq. (4.2) can be explicitly carried out. This corresponds to the so-called mean-field approximation, where all higher order Green's functions of the field $\phi(\mathbf{x})$ are given in terms of the propagator $\langle\phi\phi\rangle$.

In many cases the mean-field approximation is, however, not sufficient. Going beyond the mean-field approximation, the functional integral in Eq. (4.2) can no longer be explicitly performed. However, we can calculate the desired Green functions by exploiting Dyson–Schwinger equation techniques. Starting from the identity

$$\int \mathcal{D}\phi\,\frac{\delta}{\delta\phi}\left(\mathrm{e}^{-S[\phi]+\int j\phi}\right) = 0 \tag{4.6}$$

we can derive, in the standard fashion, a set of Dyson–Schwinger equations (DSEs) for the Green functions $\langle\phi\phi\cdots\rangle$. This infinite tower of equations has to be truncated to get a closed system of equations, and further simplifying assumptions on the form of the interaction kernels $\gamma_n$ entering the ansatz for the vacuum wave functional, see Eqs. (4.3) and (4.5), will be required. Nevertheless, this approach allows us to go beyond Gaussian wave functionals and calculate the static Green functions $\langle\phi\phi\cdots\rangle$ in terms of the kernels $\gamma_n$. By means of these static Green functions, the vacuum expectation value of the Hamiltonian $\langle\psi|\,H\,|\psi\rangle$ is expressed in terms of the kernels $\gamma_n$, which are then found by minimizing the energy density.

In the Hamiltonian approach to quantum field theory one is not primarily interested in the generating functional Eq. (4.2) itself but in expectation values of observables, in particular of the Hamiltonian. For this purpose it turns out to be more convenient to generalize Eq. (4.6) to

$$\int \mathcal{D}\phi\,\frac{\delta}{\delta\phi}\left(\mathrm{e}^{-S[\phi]}\,K[\phi]\right) = \int \mathcal{D}\phi\,\frac{\delta}{\delta\phi}\left(\psi^*[\phi]\,K[\phi]\,\psi[\phi]\right) = 0, \tag{4.7}$$

where $K[\phi]$ is an arbitrary functional of the underlying field $\phi$.

[1] As long as we ignore the $\theta$–vacuum of Yang–Mills theory, the vacuum wave functional can be chosen to be real, which we will assume in the present paper.

### 4.2.2 Derivation of the DSEs for the Hamiltonian approach to Yang–Mills theory in Coulomb gauge

Below, we apply the general Dyson–Schwinger approach to the Hamiltonian formulation of quantum field theory outlined above to Yang–Mills theory in Coulomb gauge (which also assumes Weyl gauge $A_0^a = 0$). Implementing the Coulomb gauge by the Faddeev–Popov method, the expectation value of a functional $K[A]$ of the (spatial components of the) gauge field $A$ is given by

$$\langle K[A]\rangle = \int_\Omega \mathcal{D}A\, \mathcal{J}_A\, |\psi[A]|^2\, K[A]. \tag{4.8}$$

Here, $\psi[A] = \langle A|\psi\rangle$ denotes the Yang–Mills vacuum wave functional restricted to transverse fields, $\partial_i A_i^a = 0$, and $\mathcal{J}_A = \mathrm{Det}(G_A^{-1})$ is the Faddeev–Popov determinant with

$$G_A^{-1} = (-\delta^{ab}\,\partial_{\mathbf{x}}^2 - g\,\hat{A}_i^{ab}(\mathbf{x})\partial_i^{\mathbf{x}})\delta(\mathbf{x}-\mathbf{y}) \tag{4.9}$$

being the Faddeev–Popov operator. Since we work only with spatial vectors, we will use only Lorentz subscripts. Furthermore, $g$ is the coupling constant, $\hat{A}^{ab} = f^{acb}A^c$ is the gauge field in the adjoint representation of the colour group, and $f^{acb}$ are the structure constants of the $\mathfrak{su}(N_\mathrm{c})$ algebra. The functional integration in Eq. (4.8) runs over transverse field configurations and is restricted to the first Gribov region $\Omega$ or, more precisely, to the fundamental modular region [23]. Moreover, we assume that the wave functional $\psi[A]$ is properly normalized, $\langle\psi|\psi\rangle \equiv \langle 1\rangle = 1$. Writing the vacuum wave functional as in Eq. (4.3)

$$|\psi[A]|^2 =: \mathrm{e}^{-S[A]} \tag{4.10}$$

and choosing

$$K[A] = \mathrm{e}^{\int j\cdot A}, \tag{4.11}$$

Eq. (4.8) becomes the generating functional of the static Green functions of the (transverse) gauge field $A$. In the following, it will be convenient not to fix $K[A]$ to the form (4.11) but rather to let $K[A]$ be an arbitrary functional of the gauge field. Furthermore, to simplify the bookkeeping we will use the compact notation

$$A_{k_1}^{a_1}(\mathbf{x}_1) = A(1)\,, \quad A\cdot B = A(1)\,B(1) = \int \mathrm{d}^d x\, A_i^a(\mathbf{x})\, B_i^a(\mathbf{x}), \tag{4.12}$$

such that a repeated label means summation over the discrete colour and Lorentz indices along with integration over the spatial coordinates.

Consider now the following identity

$$0 = \int_\Omega \mathcal{D}A\, \frac{\delta}{\delta A(1)} \left\{ \mathcal{J}_A\, \mathrm{e}^{-S[A]}\, K[A] \right\}, \tag{4.13}$$

which holds due to the fact that the Faddeev–Popov determinant $\mathcal{J}_A$ vanishes on the Gribov horizon $\partial\Omega$, tacitly assuming that the considered functional $K[A]$ does not spoil the vanishing of $\mathcal{J}_A K[A]$ on $\partial\Omega$. Equation (4.13) with $K[A]$ given by Eq. (4.11) becomes the ordinary DSE of the usual (Lagrangian based) functional integral formulation of Yang–Mills theory in Coulomb gauge [24] when the functional integration is extended over time-dependent gauge fields $A_\mu(\mathbf{x}, t)$ and $S[A]$ is chosen as the usual classical action of Yang–Mills theory.

Working out the functional derivative in Eq. (4.13) yields the following identity

$$\left\langle \left[ \frac{\delta \ln \mathcal{J}_A}{\delta A(1)} - \frac{\delta S[A]}{\delta A(1)} \right] K[A] \right\rangle + \left\langle \frac{\delta K[A]}{\delta A(1)} \right\rangle = 0. \tag{4.14}$$

The derivative of $\ln \mathcal{J}_A$ can be written as

$$\frac{\delta \ln \mathcal{J}_A}{\delta A(1)} = \frac{\delta}{\delta A(1)} \operatorname{Tr} \ln G_A^{-1} = \widetilde{\Gamma}_0(1;3,2)\, G_A(2,3), \tag{4.15}$$

where we have introduced the bare ghost-gluon vertex[2]

$$\widetilde{\Gamma}_0(1;2,3) = \frac{\delta G_A^{-1}(2,3)}{\delta A(1)}\,. \tag{4.16}$$

With this result, Eq. (4.14) can be cast in the form

$$\left\langle \frac{\delta S[A]}{\delta A(1)}\, K[A] \right\rangle = \left\langle \frac{\delta K[A]}{\delta A(1)} \right\rangle + \widetilde{\Gamma}_0(1;3,2)\, \langle G_A(2,3)\, K[A] \rangle\,, \tag{4.17}$$

which is the basis of the gluon DSEs, exploited below in the evaluation of $\langle H \rangle$.

Introducing ghost fields in the usual way

$$\mathcal{J}_A = \operatorname{Det}(G_A^{-1}) = \int \mathcal{D}\bar{c}\, \mathcal{D}c\, \mathrm{e}^{-\bar{c} G_A^{-1} c}, \tag{4.18}$$

the expectation value (4.8) explicitly reads

$$\langle K[A] \rangle = \int_\Omega \mathcal{D}A \int \mathcal{D}\bar{c}\, \mathcal{D}c\, K[A]\, \mathrm{e}^{-S[A] - \bar{c} G_A^{-1} c}, \tag{4.19}$$

and Eq. (4.17) can be written as

$$\left\langle \frac{\delta S[A]}{\delta A(1)}\, K[A] \right\rangle = \left\langle \frac{\delta K[A]}{\delta A(1)} \right\rangle + \widetilde{\Gamma}_0(1;3,2)\, \langle c(2)\, \bar{c}(3)\, K[A] \rangle\,. \tag{4.20}$$

The bare vertex $\widetilde{\Gamma}_0$ is the lowest-order perturbative contribution [7] to the full ghost-gluon vertex $\widetilde{\Gamma}$ defined by

$$\langle A(1)\, G_A(2,3) \rangle = \langle A(1)\, c(2)\, \bar{c}(3) \rangle = -D(1,1')\, G(2,2')\, \widetilde{\Gamma}(1';2',3')\, G(3',3), \tag{4.21}$$

where

$$D(1,2) = \langle A(1)\, A(2) \rangle \tag{4.22}$$

is the gluon propagator and

$$G(1,2) := \langle G_A(1,2) \rangle = \langle c(1)\, \bar{c}(2) \rangle\,. \tag{4.23}$$

is the ghost propagator.

Equation (4.20) [or equivalently Eq. (4.17)] is the basic DSE of the Hamiltonian formulation of Yang–Mills theory in Coulomb gauge, and we will refer to it as 'Hamiltonian DSE'. Below we will exploit this equation to express the various static (equal-time) correlators occurring in the vacuum expectation value of the Hamilton operator by the variational kernels $\gamma_n$ [Eq. (4.5)] of the wave functional $\psi[A]$. This requires appropriate choices of the so far arbitrary functional $K[A]$.

[2]The bare ghost-gluon vertex $\widetilde{\Gamma}_0$ defined by Eq. (4.16) differs from the one of Ref. [4] by an overall sign.

## 4.3 Expressing static correlators through propagators and proper vertex functions

Choosing the functional $K[A]$ in Eq. (4.20) as

$$K[A] = \exp\{j \cdot A + \bar{c} \cdot \eta + \bar{\eta} \cdot c\}\,, \tag{4.24}$$

where $j$ and $\bar{\eta}$, $\eta$ are the gluon and ghost sources, we obtain the generating functional of the full static (equal-time) Green functions

$$Z[j,\eta,\bar{\eta}] = \langle \exp\{jA + \bar{c}\eta + \bar{\eta}c\}\rangle =: \mathrm{e}^{W[j,\eta,\bar{\eta}]}, \tag{4.25}$$

where $W[j,\bar{\eta},\eta]$ is the generating functional of the connected Green functions

$$\begin{aligned} \left.\frac{\delta W}{\delta j(1)}\right|_{j=\bar{\eta}=\eta=0} &= \langle A(1)\rangle = 0, & \left.\frac{\delta^2 W}{\delta j(1)\delta j(2)}\right|_{j=\bar{\eta}=\eta=0} &= \langle A(1)A(2)\rangle\,, \\ \left.\frac{\delta^3 W}{\delta j(1)\delta\bar{\eta}(2)\delta\eta(3)}\right|_{j=\bar{\eta}=\eta=0} &= -\langle A(1)c(2)\bar{c}(3)\rangle\,, & &\text{etc.} \end{aligned} \tag{4.26}$$

Introducing the classical fields as[3]

$$A = \frac{\delta W}{\delta j}\,, \qquad \bar{c} = -\frac{\delta W}{\delta \eta}\,, \qquad c = \frac{\delta W}{\delta \bar{\eta}}\,, \tag{4.27}$$

we can define the effective action $\Gamma[A,\bar{c},c]$ through the Legendre transform

$$\Gamma[A,\bar{c},c] + W[j,\eta,\bar{\eta}] = j \cdot A + \bar{c} \cdot \eta + \bar{\eta} \cdot c, \tag{4.28}$$

where the sources have to be expressed by Eqs. (4.27) in terms of the classical fields $A$, $\bar{c}$, $c$. From the effective action Eq. (4.28), the sources are obtained as

$$j = \frac{\delta\Gamma}{\delta A}\,, \qquad \eta = \frac{\delta\Gamma}{\delta\bar{c}}\,, \qquad \bar{\eta} = -\frac{\delta\Gamma}{\delta c}\,. \tag{4.29}$$

Using Eqs. (4.27), differentiation with respect to the gluonic source can be expressed as

$$\begin{aligned} \frac{\delta}{\delta j(1)} &= \frac{\delta A(2)}{\delta j(1)}\frac{\delta}{\delta A(2)} + \frac{\delta c(2)}{\delta j(1)}\frac{\delta}{\delta c(2)} + \frac{\delta \bar{c}(2)}{\delta j(1)}\frac{\delta}{\delta \bar{c}(2)} \\ &= \frac{\delta^2 W}{\delta j(1)\delta j(2)}\frac{\delta}{\delta A(2)} + \frac{\delta^2 W}{\delta j(1)\delta\bar{\eta}(2)}\frac{\delta}{\delta c(2)} - \frac{\delta^2 W}{\delta j(1)\delta\eta(2)}\frac{\delta}{\delta\bar{c}(2)}\,. \end{aligned} \tag{4.30}$$

Similar expressions can be written for the derivatives with respect to the ghost sources. Differentiating Eqs. (4.29) and using Eqs. (4.27) and (4.30) we can link, in the usual way, the connected Green functions (derivatives of $W$) with the proper vertex functions (derivatives of $\Gamma$). As an example, we explicitly show how to express the full ghost-gluon

[3] With a slight abuse of notation, we employ the same symbol for both the classical fields and the quantum fields which are integrated over. No confusion should arise, since they never appear together. Furthermore, derivatives with respect to Grassmann fields are always left derivatives.

vertex $\widetilde{\Gamma}(1;2,3)$, defined in Eq. (4.21), by derivatives of the effective action. We start from the identity

$$\begin{aligned}\delta(1,2) &= \frac{\delta}{\delta\eta(1)}\frac{\delta\Gamma}{\delta\bar{c}(2)} \\ &= \frac{\delta^2 W}{\delta\eta(1)\delta j(2')}\frac{\delta^2\Gamma}{\delta A(2')\delta\bar{c}(2)} + \frac{\delta^2 W}{\delta\eta(1)\delta\bar{\eta}(2')}\frac{\delta^2\Gamma}{\delta c(2')\delta\bar{c}(2)} - \frac{\delta^2 W}{\delta\eta(1)\delta\eta(2')}\frac{\delta^2\Gamma}{\delta\bar{c}(2')\delta\bar{c}(2)}\,,\end{aligned} \tag{4.31}$$

which can be derived along the line of Eq. (4.30). Differentiating Eq. (4.31) with respect to a gluonic source $j(3)$ and using Eq. (4.30) yields

$$0 = \frac{\delta^3 W}{\delta j(3)\delta\eta(1)\delta\bar{\eta}(2')}\frac{\delta^2\Gamma}{\delta c(2')\delta\bar{c}(2)} + \frac{\delta^2 W}{\delta\eta(1)\delta\bar{\eta}(2')}\frac{\delta^2 W}{\delta j(3)\delta j(3')}\frac{\delta^3\Gamma}{\delta A(3')\delta c(2')\delta\bar{c}(2)} + \dots \tag{4.32}$$

where the omitted terms vanish when the sources are set to zero. By means of Eq. (4.26) and of

$$\left.\frac{\delta^2\Gamma[A,\bar{c},c]}{\delta c(2)\delta\bar{c}(1)}\right|_{A=\bar{c}=c=0} = G^{-1}(1,2), \tag{4.33}$$

the last relation can be expressed as

$$0 = -G^{-1}(2,2')\left\langle A(3)\bar{c}(1)c(2')\right\rangle + G(2',1)D(3,3')\left.\frac{\delta^3\Gamma[A,\bar{c},c]}{\delta c(2')\,\delta\bar{c}(2)\,\delta A(3')}\right|_{A=\bar{c}=c=0}. \tag{4.34}$$

Comparison with Eq. (4.21) shows

$$\widetilde{\Gamma}(1;2,3) = \left.\frac{\delta^3\Gamma[A,\bar{c},c]}{\delta c(3)\,\delta\bar{c}(2)\,\delta A(1)}\right|_{A=\bar{c}=c=0}. \tag{4.35}$$

Similarly, defining the $n$-gluon proper vertex function by

$$\Gamma_n \equiv \Gamma(1,2,\dots,n) = \left.\frac{\delta^n\Gamma[A,\bar{c},c]}{\delta A(1)\,\delta A(2)\cdots\delta A(n)}\right|_{A=\bar{c}=c=0}, \tag{4.36}$$

the full gluon $n = 2,3,4,5$-point functions defined by Eq. (4.8) with $K[A] = AA\dots$ can be expressed through the proper vertex functions as

$$D(1,2) \equiv \langle A(1)\,A(2)\rangle = \Gamma(1,2)^{-1}, \tag{4.37}$$

$$\langle A(1)\,A(2)\,A(3)\rangle = -\Gamma(1',2',3')\,D(1',1)\,D(2',2)\,D(3',3), \tag{4.38}$$

$$\begin{aligned}&\langle A(1)\,A(2)\,A(3)\,A(4)\rangle = D(1,2)D(3,4) + D(1,3)D(2,4) + D(1,4)D(2,3) \\ &+ D(1',1)\,D(2',2)\,D(3',3)\,D(4',4)\Big\{-\Gamma(1',2',3',4') \\ &+ D(5,5')\Big[\Gamma(1',2',5)\Gamma(5',3',4') + \Gamma(1',3',5)\Gamma(5',2',4') + \Gamma(1',4',5)\Gamma(5',2',3')\Big]\Big\},\end{aligned} \tag{4.39}$$

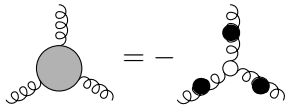


Figure 4.1: Expression of the gluon three-point function [Eq. (4.38)] by means of three-point proper vertex function and propagators. Here and in the following, fat shaded gray blobs represent full Green's functions, small filled dots connected Green's functions, and small empty dots proper vertex functions.

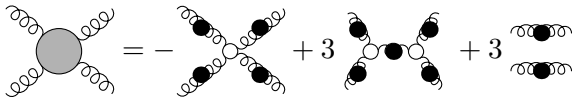


Figure 4.2: Expression of the full gluon four-point function [Eq. (4.39)]. The prefactors indicate the number of possible permutations.

$$
\begin{aligned}
\langle A(1)\,A(2)\,A(3)\,A(4)\,A(5)\rangle \;=\; & -\Gamma(1',\ldots,5')\,D(1,1')\ldots D(5,5')+ \\
& +\big[\langle A(1)A(2)A(6)\rangle\,\Gamma(6,3',4',5')D(3,3')D(4,4')D(5,5')+9\text{ combinations}\big] \\
& -\big[D(1,1')\,\langle A(1')A(6)A(7)\rangle\,\Gamma(6,6')\Gamma(7,7')\,\langle A(6')A(2)A(3)\rangle\,\langle A(7')A(4)A(5)\rangle \\
& \qquad +14\text{ combinations}\big] \\
& -\big[D(1,2)\,\langle A(3)\,A(4)\,A(5)\rangle+9\text{ combinations}\big],
\end{aligned}
\tag{4.40}
$$

The proper $n$-point gluonic functions $\Gamma(1,2,\ldots,n)$ Eq. (4.36) are by definition invariant with respect to a permutation of external legs, i.e. of the entries $1,2,\ldots,n$. Eqs. (4.38)–(4.40) are represented in diagrammatic form in Figs. 4.1–4.3. The prefactors in Figs. 4.2, 4.3 indicate the number of the possible combinations. Consider, e.g., the second diagram on the right-hand side of Fig. 4.3: out of the five external legs one can form $\binom{5}{2}=10$ pairs of external legs attached to the right vertex. The remaining three external legs have to be attached then to the left vertex and thus do not add more possible combinations. In the third diagram there are five possibilities to select the external leg attached to the internal lines. From the remaining 4 external legs there are $\binom{4}{2}=6$ possibilities to choose the two external legs at the right external vertex. This fixes also the external legs at the left vertex. The symmetry of the diagram with respect to the interchange of the two external vertices introduces an extra factor $\frac{1}{2}$. Therefore this diagram occurs with the multiplicity $5\cdot 6\cdot\frac{1}{2}=15$.

In a similar fashion one finds for the expectation value of two gauge fields and a ghost

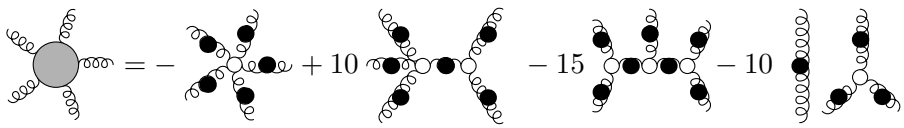


Figure 4.3: Expression for the full gluon five-point Green function [Eq. (4.40)].

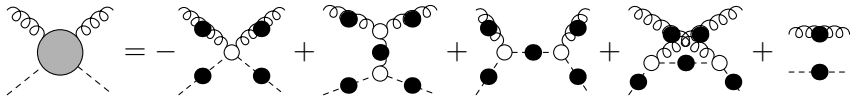


Figure 4.4: Vacuum expectation value of two gauge fields and a ghost and an anti-ghost field after Eq. (4.41).

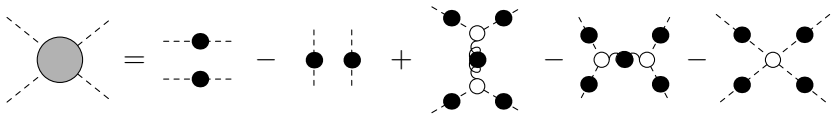


Figure 4.5: Vacuum expectation value of two ghost and two anti-ghost fields [Eq. (4.44)].

and anti-ghost field

$$\begin{aligned}\langle A(1)\,A(2)\,c(3)\,\bar{c}(4)\rangle &= \langle A(1)\,A(2)\,G_A(3,4)\rangle = D(1,2)\,G(3,4)\\ &+ D(1',1)\,D(2',2)\,G(3,3')\,G(4',4)\Big\{-\widetilde{\Gamma}(1',2';3',4')\\ &+ \Gamma(1',2',5)D(5,5')\widetilde{\Gamma}(5';3',4')\\ &+ \widetilde{\Gamma}(1';3',5)G(5,5')\widetilde{\Gamma}(2';5',4') + \widetilde{\Gamma}(2';3',5)G(5,5')\widetilde{\Gamma}(1';5',4')\Big]\Big\},\end{aligned} \tag{4.41}$$

where the two-gluon-two-ghost vertex is defined by

$$\widetilde{\Gamma}(1,2;3,4) = \left.\frac{\delta^4\Gamma[A,\bar{c},c]}{\delta c(4)\,\delta\bar{c}(3)\,\delta A(2)\,\delta A(1)}\right|_{A=\bar{c}=c=0}. \tag{4.42}$$

The diagrammatic representation of Eq. (4.41) is shown in Fig. 4.4.

The last four-point function we need for the evaluation of $\langle H\rangle$ is the ghost four-point function

$$\langle c(1)\,\bar{c}(2)\,c(3)\,\bar{c}(4)\rangle = \left\langle \left[G_A(1,2)\,G_A(3,4) - G_A(1,4)\,G_A(3,2)\right]\right\rangle, \tag{4.43}$$

which can be expressed in terms of propagators and proper vertices in the standard way, yielding

$$\begin{aligned}\langle c(1)\,\bar{c}(2)\,c(3)\,\bar{c}(4)\rangle &= G(1,2)\,G(3,4) - G(1,4)\,G(3,2)+\\ &+ G(1,1')\,G(3,3')\,\widetilde{\Gamma}(5;1',2')D(5,5')\widetilde{\Gamma}(5';3',4')\times\\ &\times [G(2',2)\,G(4',4) - G(2',4)\,G(4',2)]+\\ &- \widetilde{\Gamma}(1',3',2',4')\,G(1,1')\,G(2',2)\,G(3,3')\,G(4',4)\,,\end{aligned} \tag{4.44}$$

where the four-ghost vertex is defined by

$$\widetilde{\Gamma}(1,3,2,4) = \left.\frac{\delta^4\Gamma[A,\bar{c},c]}{\delta c(4)\,\delta c(2)\,\delta\bar{c}(3)\,\delta\bar{c}(1)}\right|_{A=\bar{c}=c=0}. \tag{4.45}$$

Equation (4.44) is represented diagrammatically in Fig. 4.5

## 4.4 The vacuum wave functional and corresponding DSEs

So far, all manipulations have been exact. In Sec. 4.2.2 we have presented the Hamiltonian DSEs for arbitrary wave functionals. To proceed further, we have to make an ansatz for the form of the vacuum wave functional $\psi[A]$, which by Eq. (4.10) defines the 'action' functional $S[A]$.

In perturbation theory, the vacuum wave functional in the form Eqs. (4.3), (4.5) has been determined up to order $\mathcal{O}(g^2)$ by a solution of the Schrödinger equation, and the resulting expressions for the kernels $\gamma_2$, $\gamma_3$, and $\gamma_4$ are given in Ref. [25]. In the present non-perturbative approach, we will assume a wave functional of the form Eqs. (4.3), (4.5) with an 'action' functional to be given by

$$\begin{aligned} S[A] = \omega(1,2)\, A(1)\, A(2) + \frac{1}{3!}\,\gamma(1,2,3)\, A(1)\, A(2)\, A(3) + \\ + \frac{1}{4!}\,\gamma(1,2,3,4)\, A(1)\, A(2)\, A(3)\, A(4). \end{aligned} \tag{4.46}$$

For historical reason, we have denoted $\frac{1}{2}\gamma(1,2)$ by $\omega(1,2)$. Consistent with our convention on the proper vertices Eq. (4.36), we will frequently use the shorthand $\gamma_n \equiv \gamma(1,\ldots,n)$. The functions $\omega \equiv \frac{1}{2}\gamma_2$, $\gamma_3$, and $\gamma_4$ are variational kernels which will be determined by minimization of the vacuum energy density. As discussed before, Eq. (4.46) can be considered as arising in leading orders of a systematic Taylor expansion of the 'action' functional.

Let us also stress that the ghost fields do not enter the Yang–Mills vacuum wave functional $\psi[A]$. The ghost fields are auxiliary fields to represent the Faddeev–Popov determinant in local action form. By the very definition of the ghost fields [Eq. (4.18)] the ghost-gluon vertex in the 'action' [i.e. the exponent of Eq. (4.19)] has to be the bare vertex and, in principle, there is absolutely no need to include ghost or ghost-gluon kernels as variational kernels in the wave functional. However, due to approximations to be introduced, for practical purposes, one might also include ghost or ghost-gluon vertices as variational kernels in the wave functional to improve the latter. If the exact gluon wave functional $\psi[A]$ were used, the variational principle would determine the ghost-gluon kernel as the bare one and higher ghost kernels to vanish. Therefore we will not include additional ghost vertices into the variational ansatz for the vacuum wave functional.

By construction, the variational kernels $\gamma_n$ (which are purely gluonic) are totally symmetric with respect to permutations of the overall indices. Furthermore, the wave functional defined by Eqs. (4.10), (4.46) is normalizable even when the restriction of the functional integration to the first Gribov region is ignored, provided the kernel $\gamma_4$ is positive definite, which we will assume for the moment and which later on will be confirmed by our calculations. With the action functional Eq. (4.46), the Hamiltonian DSE (4.17) becomes

$$\begin{aligned} 2\omega(1,2)\,\langle A(2)K[A]\rangle + \frac{1}{2}\,\gamma(1,2,3)\,\langle A(2)\,A(3)\,K[A]\rangle + \\ + \frac{1}{3!}\,\gamma(1,2,3,4)\,\langle A(2)\,A(3)\,A(4)\,K[A]\rangle = \\ = \left\langle \frac{\delta K[A]}{\delta A(1)} \right\rangle + \widetilde{\Gamma}_0(1;3,2)\,\langle G_A(2,3)\,K[A]\rangle\,. \end{aligned} \tag{4.47}$$

Except for the non-locality of the variational kernels $\omega$, $\gamma_3$, and $\gamma_4$, the functional Eq. (4.46) has the same structure as the ordinary Yang–Mills action. Therefore, the DSEs

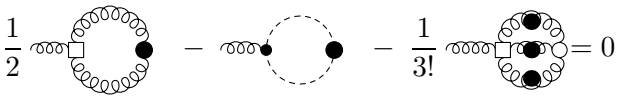


Figure 4.6: Diagrammatic representation of Eq. (4.48). The empty square boxes denote the variational kernels $\gamma_n$.

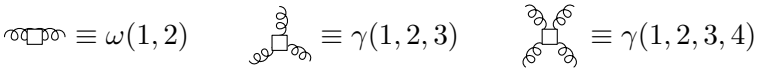


Figure 4.7: Variational kernels occurring in the exponent of the wave functional.

resulting from Eq. (4.47) will have the same structure as the DSEs of ordinary $d = 3$ Yang–Mills theory in Landau gauge, however with bare vertices replaced by the non-local variational kernels $\omega$, $\gamma_3$, and $\gamma_4$. Equation (4.47) is our fundamental DSE for the Hamiltonian approach to Yang–Mills theory in Coulomb gauge.

### 4.4.1 DSEs of gluonic vertex functions

The first DSE is obtained by setting $K[A] = 1$ in Eq. (4.47). Using $\langle A \rangle = 0$ and the expression Eq. (4.38) for the three-point function yields the identity

$$\begin{aligned} 0 = {} & \frac{1}{2}\,\gamma(1,2,3) D(2,3) - \widetilde{\Gamma}_0(1;3,2) G(2,3) \\ & - \frac{1}{3!}\,\gamma(1,2,3,4)\,\Gamma(2',3',4')\,D(2,2')\,D(3,3')\,D(4,4'), \end{aligned} \tag{4.48}$$

which is diagrammatically illustrated in Fig. 4.6. Equation (4.48) is not really a dynamical equation but rather a constraint, which can be used to simplify tadpole terms in the evaluation of higher-order DSEs. It is also easy to see that in lowest order perturbation theory each term in Eq. (4.48) vanishes separately.

The DSE for the gluon propagator follows from Eq. (4.47) by putting $K[A] = A$, yielding

$$\begin{aligned} & 2\omega(1,3)\,\langle A(3)\,A(2)\rangle + \frac{1}{2}\,\gamma(1,3,4)\,\langle A(3)\,A(4)\,A(2)\rangle + \\ & + \frac{1}{3!}\,\gamma(1,3,4,5)\,\langle A(3)\,A(4)\,A(5)\,A(2)\rangle = t(1,2) + \widetilde{\Gamma}_0(1;4,3)\,\langle G_A(3,4)\,A(2)\rangle\,, \end{aligned} \tag{4.49}$$

where we have introduced the abbreviation

$$t(1,2) \equiv \delta^{a_1 a_2}\, t_{k_1 k_2}(\mathbf{x}_1)\,\delta(\mathbf{x}_1 - \mathbf{x}_2) \tag{4.50}$$

and $t_{k_1 k_2}(\mathbf{x}_1) = \delta_{k_1 k_2} - \partial^{x_1}_{k_1}\partial^{x_1}_{k_2}/\partial^2_{x_1}$ is the transverse projector. By means of Eqs. (4.38), (4.39), the three- and four-point functions in Eq. (4.49) can be expressed through the proper vertex functions $\Gamma_n$. By multiplying Eq. (4.49) by the inverse gluon propagator Eq. (4.37) and defining

$$D(1,2)^{-1} = \Gamma(1,2) =: 2\,\Omega(1,2), \tag{4.51}$$

Eq. (4.49) can be cast in the form

$$\Omega(1,2) = \omega(1,2) - \xi(1,2) + \chi(1,2) + \phi_1(1,2) - \phi_2(1,2) + \phi_t(1,2), \tag{4.52}$$

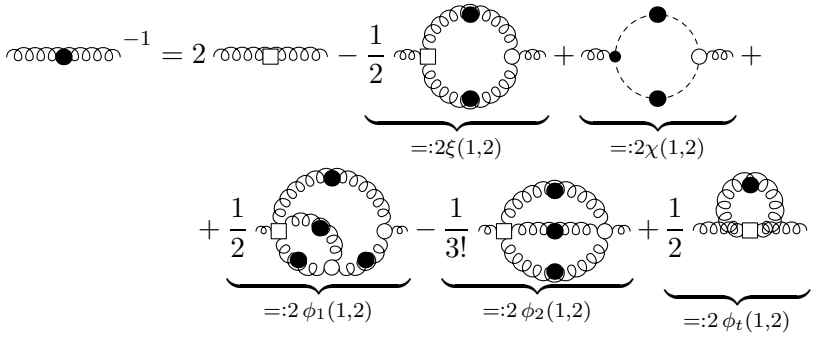



Figure 4.8: Diagrammatic representation of the DSE for the gluon propagator, Eq. (4.52).

where we have introduced the following loop terms

$$\xi(1,2) = \tfrac{1}{4}\,\gamma(1,3,4)\,D(3,3')\,D(4,4')\,\Gamma(3',4',2), \tag{4.53a}$$
$$\chi(1,2) = \tfrac{1}{2}\,\widetilde{\Gamma}_0(1;3,4)\,G(3',3)\,G(4,4')\,\widetilde{\Gamma}(2;4',3'), \tag{4.53b}$$
$$\phi_1(1,2) = \tfrac{1}{4}\,\gamma(1,3,4,5)\,D(3,3')\,D(4,4')\,D(5,5')\,D(6,6')\,\Gamma(4',5',6)\,\Gamma(3',6',2), \tag{4.53c}$$
$$\phi_2(1,2) = \tfrac{1}{3!2}\,\gamma(1,3,4,5)\,D(3,3')\,D(4,4')\,D(5,5')\,\Gamma(3',4',5',2), \tag{4.53d}$$
$$\phi_t(1,2) = \tfrac{1}{4}\,\gamma(1,2,3,4)\,D(3,4). \tag{4.53e}$$

Equation (4.52) is represented diagrammatically in Fig. 4.8, and is recognized as the usual DSE for the gluon propagator of Landau gauge Yang–Mills theory [26], except for the replacement of the bare Yang–Mills vertices (defined by the Yang–Mills Lagrangian) by the variational kernels $\gamma_n$ (defined by the ansatz Eq. (4.46) for the vacuum functional), which are represented by open square boxes, see Fig. 4.7. Note that the gluon loop $\xi(1,2)$ [Eq. (4.53a)] disappears when the three-gluon kernel $\gamma_3$ is absent from the exponential [Eq. (4.46)] of the wave functional [Eq. (4.10)]. For a Gaussian wave functional ($\gamma_3 = \gamma_4 = 0$) only the ghost loop $\chi(1,2)$ [Eq. (4.53b)] survives from the loop terms in the DSE (4.52), Fig. 4.8.

Choosing $K[A] = A(2)A(3)$ in Eq. (4.47) yields the DSE for the three-gluon vertex,

$$\begin{aligned} &2\omega(1,4)\,\langle A(4)A(2)A(3)\rangle + \frac{1}{2}\,\gamma(1,4,5)\,\langle A(4)\,A(5)\,A(2)\,A(3)\rangle + \\ &+ \frac{1}{3!}\,\gamma(1,4,5,6)\,\langle A(4)\,A(5)\,A(6)\,A(2)\,A(3)\rangle = \widetilde{\Gamma}_0(1;5,4)\,\langle G_A(4,5)\,A(2)\,A(3)\rangle\,. \end{aligned} \tag{4.54}$$

By means of Eqs. (4.38)–(4.41), the first two terms on the left-hand side and the right-hand side of Eq. (4.54) can be expressed in terms of proper vertex functions $\Gamma_n$. The explicit evaluation of the five-point function is quite lengthy, and we quote only the result. Restricting ourselves to terms involving up to one loop (which is sufficient to obtain the energy $\langle H\rangle$ up to three overlapping loops, see the introduction), and chopping off the external propagators we eventually find from Eq. (4.54) the DSE for the proper three-point vertex function $\Gamma(1,2,3)$

$$\Gamma(1,2,3) = \gamma(1,2,3) + \gamma(1,4,5)\,D(4,4')\,D(5,5')\,D(6,6')\,\Gamma(2,4',6)\,\Gamma(3,5',6')$$

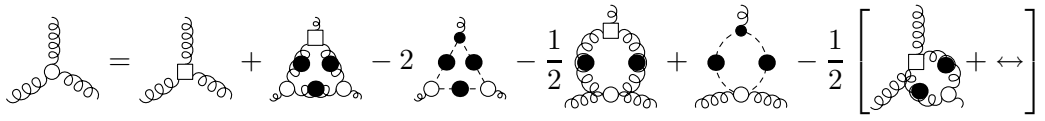


Figure 4.9: Diagrammatic representation of the DSE (4.55) for the three-gluon proper vertex function. The factor 2 in front of the ghost loop accounts for the two diagrams differing in the direction of the ghost line.

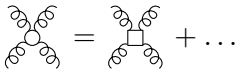


Figure 4.10: Leading order DSE for the four-gluon vertex.

$$
\begin{aligned}
&- \widetilde{\Gamma}_0(1;4,5)\, G(4',4)\, G(5,5')\, G(6',6) \big[\widetilde{\Gamma}(2;6,4')\, \widetilde{\Gamma}(3;5',6') + 2 \leftrightarrow 3\big] \\
&- \frac{1}{2}\, \gamma(1,4,5)\, D(4,4')\, D(5,5')\, \Gamma(4',5',2,3) \\
&+ \widetilde{\Gamma}_0(1;4,5)\, G(4',4)\, G(5,5')\, \widetilde{\Gamma}(2,3;5',4') \\
&- \frac{1}{2} \big[\gamma(1,2,4,5)\, D(4,4')\, D(5,5')\, \Gamma(4',5',3) + 2 \leftrightarrow 3\big], \qquad (4.55)
\end{aligned}
$$

which is represented diagrammatically in Fig. 4.9.

Analogously, one can derive the DSE for the four-gluon vertex. Restricting ourselves again up to three loops in the energy $\langle H \rangle$, one finds just the "tree-level" expression

$$
\Gamma(1,2,3,4) = \gamma(1,2,3,4) + \ldots \qquad (4.56)
$$

Any loop contribution to $\Gamma_4$ generates at least four-loop terms in the energy, which are beyond the scope of the present paper.

### 4.4.2 The DSEs for the ghost propagator and the ghost-gluon vertex

Inverting the defining equation of the Faddeev–Popov operator [Eq. (4.9)] one finds the following identity [4]

$$
G_A(1,2) = G_0(1,2) - G_A(1,4)\, A(3)\, \widetilde{\Gamma}_0(3;4,5)\, G_0(5,2), \qquad (4.57)
$$

where $G_0(1,2) = [(-\partial^2)^{-1}](1,2)$ is the bare ghost propagator and $\widetilde{\Gamma}_0$ is the bare ghost-gluon vertex defined in Eq. (4.16). Taking the expectation value of Eq. (4.57) and using Eq. (4.21) yields the usual DSE for the ghost propagator [4]

$$
G(1,2)^{-1} = G_0(1,2)^{-1} - \widetilde{\Gamma}(3;1,4)\, G(4,4')\, D(3,3')\, \widetilde{\Gamma}_0(3';4',2), \qquad (4.58)
$$

which in momentum space reads

$$
G^{-1}(\mathbf{p}) = \mathbf{p}^2 + \frac{\mathrm{i}\, g}{N_\mathrm{c}^2 - 1} \int \frac{\mathrm{d}^d q}{(2\pi)^d}\, f^{abc} \widetilde{\Gamma}_i^{abc}(\mathbf{q}; \mathbf{p}-\mathbf{q}, -\mathbf{p})\, \frac{t_{ij}(\mathbf{q})\, p_j}{2\Omega(\mathbf{q})}\, G(\mathbf{p}-\mathbf{q}), \qquad (4.59)
$$

and which is represented diagrammatically in Fig. 4.11.

Figure 4.11: Diagrammatic representation of the ghost DSE Eq. (4.58).

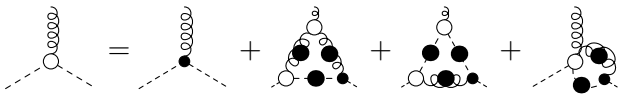


Figure 4.12: Diagrammatic representation of the DSE (4.60) for the ghost-gluon vertex.

To derive the DSE for the ghost-gluon vertex there are two possibilities. The first one is to multiply Eq. (4.57) by the gauge field and to take the expectation value of the resulting expression. This leads to

$$\langle A(1)\, G_A(2,3)\rangle = -\,\langle A(1)\, A(4)\, G_A(2,5)\rangle\; \widetilde{\Gamma}_0(4;5,6)\, G_0(6,3). \tag{4.60}$$

The remaining expectation value can be expressed in terms of proper vertices by means of Eqs. (4.21) and (4.41). After chopping off the external propagators, this results in the DSE for the ghost-gluon vertex shown in Fig. 4.12. Equation (4.60) is exact, i.e. not truncated, but not very convenient for the evaluation of the energy density. A more convenient form of the DSE for the ghost-gluon vertex is obtained by putting $K[A] = G_A$ in our general Hamiltonian DSE (4.47), thereby using the chain rule for the derivative of the ghost Green's function and using Eqs. (4.41), (4.44) to express vacuum expectation values through propagators and proper vertex functions. The resulting equation reads at one-loop level

$$\begin{aligned}\widetilde{\Gamma}(1;2,3) = {}& \widetilde{\Gamma}_0(1;2,3) + \gamma(1,4,5)\, D(4,4')\, D(5,5')\, G(6,6')\, \widetilde{\Gamma}(4';2,6)\, \widetilde{\Gamma}(5';6',3) \\ & + \widetilde{\Gamma}_0(1;4,5)\, G(4',4)\, G(5,5')\, D(6,6')\, \widetilde{\Gamma}(6,2,4')\, \widetilde{\Gamma}(6';5',3) \\ & - \frac{1}{2}\, \gamma(1,4,5)\, D(4,4')\, D(5,5')\, \Gamma(4',5';2,3) \\ & + \widetilde{\Gamma}(1;4,5)\, G(4',4)\, G(5,5')\, \widetilde{\Gamma}(2,5',3,4').\end{aligned} \tag{4.61}$$

and is shown in Fig. 4.13. The main difference between Eqs. (4.60) and (4.61) is that while Eq. (4.60) is exact, in Eq. (4.61) two-loops terms involving higher-order vertices are neglected. Nevertheless, to our purpose, calculating the energy up to three loops, Eq. (4.61) is more convenient. We will use this equation in Sec. 4.5 to simplify the expression for the kinetic and Coulomb energy.

## 4.5 Energy density of the Yang–Mills vacuum

The DSEs of the Hamiltonian approach derived in Sec. 4.4 are not 'equations of motion' in the usual sense, but rather connect the various Green functions with the kernels occurring

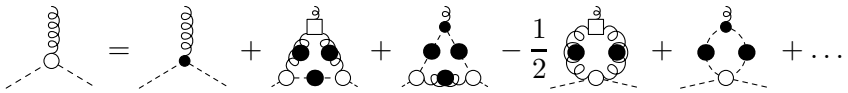


Figure 4.13: Diagrammatic representation of the DSE (4.61) for the ghost-gluon vertex.

in the ansatz for the wave functional, while these kernels themselves are at this point not yet fixed. Here is where the variational principle comes into play: we will now evaluate the expectation value of the Yang–Mills Hamiltonian for the wave functional (4.10) with the ansatz (4.46) and then minimize it with respect to the variational kernels $\gamma_n$.

The Yang–Mills Hamiltonian in Coulomb gauge reads [27]

$$\begin{aligned} H = & \int \mathrm{d}^d x \left[ \frac{1}{2} \mathcal{J}_A^{-1} \Pi_i^a(\mathbf{x}) \mathcal{J}_A \Pi_i^a(\mathbf{x}) + \frac{1}{4} F_{ij}^a(\mathbf{x}) F_{ij}^a(\mathbf{x}) \right] \\ & + \frac{g^2}{2} \int \mathrm{d}^d x \; \mathrm{d}^d y \; \mathcal{J}_A^{-1} \rho^a(\mathbf{x}) F_A^{ab}(\mathbf{x}, \mathbf{y}) \mathcal{J}_A \rho^b(\mathbf{y}). \end{aligned} \tag{4.62}$$

Here, $\Pi_k^a(\mathbf{x}) = -\mathrm{i}\delta/\delta A_k^a(\mathbf{x})$ is the momentum operator, and $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + g f^{abc} A_i^b A_j^c$, is the non-abelian field strength tensor. The first two terms in Eq. (4.62) are the electric (kinetic) and magnetic parts of the ordinary Yang–Mills Hamiltonian restricted to the curvilinear "coordinate" space of Coulomb gauge. The third (Coulomb) term arises from the resolution of Gauss's law and describes the interaction of the non-abelian colour charge (of the fluctuating gauge field) with density

$$\rho^a(\mathbf{x}) = \hat{A}_i^{ab}(\mathbf{x}) \Pi_i^b(\mathbf{x}) \tag{4.63}$$

through the non-abelian Coulomb interaction kernel

$$F_A^{ab}(\mathbf{x}, \mathbf{y}) = \left[ (-\hat{D}\partial)^{-1} (-\partial^2) (-\hat{D}\partial)^{-1} \right]_{\mathbf{x},\mathbf{y}}^{a,b}. \tag{4.64}$$

The Yang–Mills Hamiltonian in Coulomb gauge Eq. (4.62) is a positive definite operator. Accordingly the energy $\langle \psi | H | \psi \rangle$ is bounded from below (by zero) and the variational principle is applicable.

For later use, we rewrite the magnetic term of the Hamiltonian in the symmetrized form

$$\frac{1}{4} F_{ij}^a F_{ij}^a = -\frac{1}{2} A \partial^2 A + \frac{g}{3!} T_3 A^3 + \frac{g^2}{4!} T_4 A^4, \tag{4.65}$$

where the interaction kernels are given in momentum space by

$$T_{ijk}^{abc}(\mathbf{p}, \mathbf{q}, \mathbf{k}) = \mathrm{i} f^{abc} [\delta_{ij}(p-q)_k + \delta_{jk}(q-k)_i + \delta_{ki}(k-p)_j] \tag{4.66a}$$

and

$$\begin{aligned} T_{ijkl}^{abcd} = \big\{ & f^{abe} f^{cde} (\delta_{ik}\, \delta_{jl} - \delta_{il}\, \delta_{jk}) \\ & + f^{ace} f^{bde} (\delta_{ij}\, \delta_{kl} - \delta_{jk}\, \delta_{il}) \\ & + f^{ade} f^{bce} (\delta_{ij}\, \delta_{kl} - \delta_{ik}\, \delta_{jl}) \big\}. \end{aligned} \tag{4.66b}$$

The bare four-gluon vertex [Eq. (4.66b)] is independent of the momenta.

### 4.5.1 Technicalities

To evaluate the vacuum expectation values of the kinetic term and of the Coulomb Hamiltonian, it is convenient to perform an integration by parts in the gauge field. This leads to expressions of the form

$$\left\langle \frac{\delta S[A]}{\delta A(1)} \frac{\delta S[A]}{\delta A(2)} f[A] \right\rangle, \tag{4.67}$$

where $f[A]$ is a functional of the gauge field which does not contain any momentum operator. In principle, we could now explicitly write down the variations of the action Eq. (4.46) and evaluate the expectation value Eq. (4.67). Then one would recognize that some terms can be combined and simplified by using the DSEs (4.52), (4.55). Therefore, a more efficient way to evaluate the expectation value Eq. (4.67) is to use the Hamiltonian DSEs from the very beginning.

Putting $K[A] = \delta S/\delta A(2) f[A]$ in the general DSE (4.17) we obtain

$$\begin{aligned}\left\langle \frac{\delta S[A]}{\delta A(1)} \frac{\delta S[A]}{\delta A(2)} f[A] \right\rangle = & \left\langle \frac{\delta^2 S[A]}{\delta A(1)\,\delta A(2)} f[A] \right\rangle \\ & + \left\langle \frac{\delta f[A]}{\delta A(1)} \frac{\delta S[A]}{\delta A(2)} \right\rangle + \widetilde{\Gamma}_0(1;3,4) \left\langle \frac{\delta S[A]}{\delta A(2)} G_A(4,3) \right\rangle .\end{aligned} \tag{4.68}$$

The last two terms on the r.h.s. of Eq. (4.68) can again be re-expressed through the DSEs (4.17), thereby putting $K[A] = \delta f[A]/\delta A$ and $K[A] = G_A$, respectively, and using the definition of the bare ghost-gluon vertex $\widetilde{\Gamma}_0$, Eq. (4.16). This results finally in the relation

$$\begin{aligned}\left\langle \frac{\delta S[A]}{\delta A(1)} \frac{\delta S[A]}{\delta A(2)} f[A] \right\rangle = & \left\langle \frac{\delta^2 S[A]}{\delta A(1)\,\delta A(2)} f[A] \right\rangle + \left\langle \frac{\delta^2 f[A]}{\delta A(1)\,\delta A(2)} \right\rangle \\ & + \widetilde{\Gamma}_0(1;4,3) \left\langle \frac{\delta f[A]}{\delta A(2)} G_A(3,4) \right\rangle + \widetilde{\Gamma}_0(2;4,3) \left\langle \frac{\delta f[A]}{\delta A(1)} G_A(3,4) \right\rangle \\ & + \widetilde{\Gamma}_0(1;4,3)\, \widetilde{\Gamma}_0(2;6,5) \left\langle f[A] \big[ G_A(3,4)\, G_A(5,6) - G_A(3,6)\, G_A(5,4) \big] \right\rangle .\end{aligned} \tag{4.69}$$

We stress that this is an exact identity, which holds for $f[A]$ being an arbitrary functional of the gauge field only, i.e. not containing the momentum operator. Furthermore, it will be sometimes convenient to express the last expectation value in terms of ghost fields

$$\left\langle f[A] \big[ G_A(3,4)\, G_A(5,6) - G_A(3,6)\, G_A(5,4) \big] \right\rangle = \langle f[A]\, c(3)\, \bar{c}(4)\, c(5)\, \bar{c}(6) \rangle . \tag{4.70}$$

### 4.5.2 Kinetic energy

After an integration by parts, the vacuum expectation value of the kinetic part of the Yang–Mills Hamiltonian [first term in Eq. (4.62)] can be expressed as

$$E_k = \frac{1}{2} \int_\Omega \mathcal{D}A\, \mathcal{J}_A \frac{\delta \psi[A]}{\delta A(1)} \frac{\delta \psi[A]}{\delta A(1)} = \frac{1}{8} \left\langle \frac{\delta S[A]}{\delta A(1)} \frac{\delta S[A]}{\delta A(1)} \right\rangle . \tag{4.71}$$

The last expectation value has precisely the form of Eq. (4.69) with $f[A] = 1$ and the two external indices contracted. The terms in Eq. (4.69) involving functional derivatives of $f[A]$ then vanish, and with the explicit form of the action Eq. (4.46) we find for the kinetic energy

$$E_k = \frac{1}{8} \Big[ 2\,\omega(1,1) + 2\,\phi_t(1,1) + \widetilde{\Gamma}_0(1;4,3)\, \widetilde{\Gamma}_0(1;6,5) \left\langle c(3)\, \bar{c}(4)\, c(5)\, \bar{c}(6) \right\rangle \Big] . \tag{4.72}$$

Here, $\phi_t$ is the gluon tadpole term occurring in the gluon DSE (4.52) and being defined by Eq. (4.53e). The ghost four-point function $\langle c\bar{c}c\bar{c} \rangle$ occurring in the last term can be expressed by means of Eq. (4.44) in terms of propagators and proper functions. Contracting

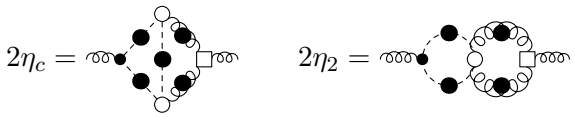


Figure 4.14: Diagrammatic representation of the contributions (4.75) to the kinetic energy density.

Eq. (4.44) with the two bare ghost-gluon vertices as in Eq. (4.72), one obtains

$$\begin{aligned}\widetilde{\Gamma}_0(1;4,3)\,\widetilde{\Gamma}_0(1;6,5)\,\langle c(3)\,\bar{c}(4)\,c(5)\,\bar{c}(6)\rangle = 4\chi(1,3)D(3,4)\chi(4,1) - \Big[\widetilde{\Gamma}_0(1;5',6') \\ + \widetilde{\Gamma}_0(1;4,3)\widetilde{\Gamma}(7;3',6')\widetilde{\Gamma}(7';5',4')G(3,3')G(4',4)D(7,7') \\ + \widetilde{\Gamma}_0(1;4,3)\widetilde{\Gamma}(3',5',4',6')G(3,3')G(4',4)\Big]\,G(6',6)\,G(5,5')\,\widetilde{\Gamma}_0(1;6,5).\end{aligned} \tag{4.73}$$

where $\chi(1,2)$ is the ghost loop defined by Eq. (4.53b). The terms in the square brackets represent precisely the first, third, and fifth term of the right-hand side of the truncated DSE (4.61) for the ghost-gluon vertex. Therefore we can use Eq. (4.61) to rewrite the terms in the bracket in Eq. (4.73) in more compact form

$$E_k = \frac{1}{4}[\omega(1,1) + \phi_t(1,1) - \chi(1,1) + 2\chi(1,2)D(2,3)\chi(3,1) + \eta_c(1,1) - \eta_2(1,1)], \tag{4.74}$$

where we have introduced the abbreviations

$$\begin{aligned}2\eta_c(1,2) &= \widetilde{\Gamma}_0(1;3,4)G(3',3)G(4,4')G(5,5')\widetilde{\Gamma}(6';5',3')\widetilde{\Gamma}(7';4',5)D(6,6')D(7,7')\gamma(6,7,2), \\ 2\eta_2(1,2) &= \widetilde{\Gamma}_0(1;3,4)G(3',3)G(4,4')\widetilde{\Gamma}(5',6';4',3')D(5,5')D(6,6')\gamma(5,6,2),\end{aligned} \tag{4.75}$$

see Fig. 4.14. Equation (4.74) can be slightly rewritten by using the gluon DSE (4.52) to eliminate the variational kernel $\omega$ (or, more precisely, the sum $\omega + \phi_t$) in favour of the inverse gluon propagator Eq. (4.51). Assuming furthermore that the various loop terms [Eqs. (4.53)] are colour diagonal, e.g.

$$\xi^{ab}_{ij}(\mathbf{k}) = \delta^{ab}\,t_{ij}(\mathbf{k})\,\xi(\mathbf{k}),\ \text{etc.} \tag{4.76}$$

which is guaranteed by global colour invariance, we can express the kinetic energy in momentum space as

$$E_k = \frac{(N_c^2-1)(d-1)}{4}\,V\int \text{đ}p\left\{\frac{[\Omega(\mathbf{p})-\chi(\mathbf{p})]^2}{\Omega(\mathbf{p})} + \xi(\mathbf{p}) - \phi_1(\mathbf{p}) + \phi_2(\mathbf{p}) - 2\eta_c(\mathbf{p})\right\}, \tag{4.77}$$

where the terms $\phi_{1,2}$ are defined in Eqs. (4.53c) and (4.53d), and $\eta_c$ is given in Eq. (4.75). Furthermore, in Eq. (4.77) we have introduced the abbreviation

$$\text{đ}p \equiv \frac{\mathrm{d}^d p}{(2\pi)^d}\,,$$

and $V$ is the spatial volume which arises as in Ref. [4].

The ghost and gluon loop contributions defined in Eqs. (4.53) read in momentum space

$$\chi(\mathbf{p}) = \frac{t_{ij}(\mathbf{p})}{2(N_c^2-1)(d-1)} \int \text{đ}q\, \widetilde{\Gamma}^{abc}_{0,i}(\mathbf{p};\mathbf{q}-\mathbf{p},-\mathbf{q})\, \widetilde{\Gamma}^{acb}_{j}(-\mathbf{p};\mathbf{q},\mathbf{p}-\mathbf{q}) G(\mathbf{q}) G(\mathbf{p}-\mathbf{q}), \tag{4.78}$$

$$\xi(\mathbf{p}) = \frac{1}{16(N_c^2-1)(d-1)} \int \text{đ}q\ \mathrm{d}k\ \frac{(\gamma_3 \circ \Gamma_3)(\mathbf{p},\mathbf{q},\mathbf{k})}{\Omega(\mathbf{q})\,\Omega(\mathbf{k})}\, \delta(\mathbf{p}+\mathbf{q}+\mathbf{k})\,. \tag{4.79}$$

In the above equations, $\widetilde{\Gamma}$ and $\Gamma_3$ are, respectively, the full ghost-gluon and three-gluon vertices defined by Eqs. (4.35), (4.36). We have also introduced here the contraction of two colour and Lorentz tensor structures through transverse projectors as

$$A \circ B := A^{abc}_{ijk}(\mathbf{p},\mathbf{q},\mathbf{k})\, t_{il}(\mathbf{p})\, t_{jm}(\mathbf{q})\, t_{kn}(\mathbf{k})\, B^{abc}_{lmn}(-\mathbf{p},-\mathbf{q},-\mathbf{k}). \tag{4.80}$$

Furthermore, the term $\eta_2$ [Eq. (4.75)] has been discarded, since it gives rise exclusively to higher-order contributions with more than three loops in the energy, which are beyond our truncation scheme.

### 4.5.3 Magnetic energy

In the notation of Eq. (4.65) the magnetic energy is given by

$$E_B = \frac{1}{4}\left\langle F_{ij}^2 \right\rangle = -\frac{1}{2}\left\langle A\partial^2 A \right\rangle + \frac{g}{3!}\, T_3 \left\langle A^3 \right\rangle + \frac{g^2}{4!}\, T_4 \left\langle A^4 \right\rangle. \tag{4.81}$$

The first term on the right-hand side of Eq. (4.81) can be expressed by means of the gluon propagator [Eq. (4.22)], while the second term can be expressed through the proper three-point function $\Gamma_3$ [Eq. (4.38)]:

$$\frac{g}{3!}\, T_3 \left\langle A^3 \right\rangle = -\frac{g}{3!}\, T_3 \circ \Gamma_3 \left\langle AA \right\rangle^3. \tag{4.82}$$

In momentum space, these two energy contributions read

$$\frac{(N_c^2-1)(d-1)}{4}\, V \int \text{đ}p\, \frac{\mathbf{p}^2}{\Omega(\mathbf{p})} - \frac{g\,V}{8\cdot 3!} \int \text{đ}p\ \text{đ}q\ \mathrm{d}k\, \frac{(T_3 \circ \Gamma_3)(\mathbf{p},\mathbf{q},\mathbf{k})}{\Omega(\mathbf{p})\,\Omega(\mathbf{q})\,\Omega(\mathbf{k})}\, \delta(\mathbf{p}+\mathbf{q}+\mathbf{k}). \tag{4.83}$$

Let us turn now to the last term in Eq. (4.81), $\left\langle A^4 \right\rangle$. The four-point function is expressed by means of Eq. (4.39) in terms of gluon propagators and proper vertex functions. The disconnected terms in Eq. (4.39), i.e. the products of two gluon propagators, when contracted with the bare four-gluon vertex $T_4$ [Eq. (4.66b)] results in

$$g^2\, \frac{N_c(N_c^2-1)}{16}\, V \int \text{đ}p\ \text{đ}q\, \frac{d(d-3)+3-(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2}{\Omega(\mathbf{p})\,\Omega(\mathbf{q})}\,. \tag{4.84}$$

which is the usual gluon tadpole term, which occurs already when a Gaussian wave functional is used. Notice that, since the $q$ integral does not depend on an external momentum, we can replace $\hat{q}_i\hat{q}_j \to \frac{1}{d}\delta_{ij}$ in the integrand, and using

$$d(d-3)+3-\frac{1}{d} = \frac{(d-1)^3}{d}$$

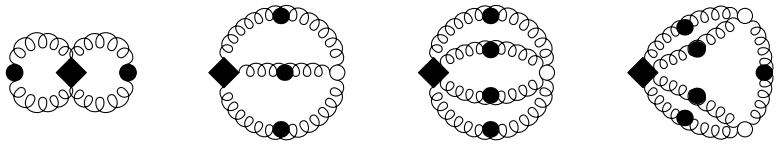


Figure 4.15: Contributions from the three- and four-gluon vertices to the magnetic energy, from left to right: second term in Eq. (4.83), Eq. (4.84), Eq. (4.85), and Eq. (4.86). The filled diamonds stand for the bare vertices $T_3$, $T_4$ [Eqs. (4.66)] occurring in the magnetic part of the Hamilton operator [Eq. (4.65)].

we can rewrite Eq. (4.84) as

$$g^2\,\frac{N_{\rm c}(N_{\rm c}^2-1)}{16}\,\frac{(d-1)^3}{d}\,V\int \text{đ}p\;\text{đ}q\,\frac{1}{\Omega(\mathbf{p})\,\Omega(\mathbf{q})}\,. \tag{4.84$'$}$$

Besides this, we get from the last term in Eq. (4.81) by using Eq. (4.39) also a contribution containing the proper four-point vertex function $\Gamma_4$:

$$-\frac{g^2\,V}{16\cdot 4!}\int \text{đ}p\;\text{đ}q\;\text{đ}k\;\text{d}\ell\frac{(T_4\circ\Gamma_4)}{\Omega(\mathbf{p})\,\Omega(\mathbf{q})\,\Omega(\mathbf{k})\,\Omega(\boldsymbol{\ell})}\,\delta(\mathbf{p}+\mathbf{q}+\mathbf{k}+\boldsymbol{\ell}), \tag{4.85}$$

and a contribution containing two three-gluon vertices

$$\frac{g^2\,V}{8}\int \text{đ}[pqk\ell]\frac{T^{abcd}_{ijmn}\,\Gamma^{abe}_{ijl}(-\mathbf{p},-\mathbf{q},\mathbf{p}+\mathbf{q})\,\Gamma^{ecd}_{lmn}(\mathbf{k}+\boldsymbol{\ell},-\mathbf{k},-\boldsymbol{\ell})}{32\Omega(\mathbf{p})\,\Omega(\mathbf{q})\,\Omega(\mathbf{k})\,\Omega(\boldsymbol{\ell})\,\Omega(\mathbf{p}+\mathbf{q})}\,(2\pi)^d\delta(\mathbf{p}+\mathbf{q}+\mathbf{k}+\boldsymbol{\ell}). \tag{4.86}$$

The Lorentz indices in Eq. (4.86) are supposed to be contracted by transverse projectors, which we have not explicitly written down in order to prevent the equation from getting cluttered. The various interaction contributions to the magnetic energy given by Eqs. (4.83)–(4.86) are shown in Fig. 4.15. Notice that the diagrams with a four-gluon vertex contain already three loops.

### 4.5.4 Coulomb energy

After an integration by parts, as in the case of the kinetic energy, the vacuum expectation value of the Coulomb Hamiltonian [last term in Eq. (4.62)] can be expressed as

$$E_c=\frac{g^2}{2}\int\mathcal{D}A\,\mathcal{J}_A\int \text{d}^dx\;\text{d}^dy\left[\hat{A}^{ac}_i(\mathbf{x})\,\frac{\delta\psi[A]}{\text{i}\,\delta A^c_i(\mathbf{x})}\right]^*F^{ab}_A(\mathbf{x},\mathbf{y})\left[\hat{A}^{bd}_j(\mathbf{y})\,\frac{\delta\psi[A]}{\text{i}\,\delta A^d_j(\mathbf{y})}\right]. \tag{4.87}$$

In order to exploit our compact notation, we rewrite the colour charge density Eq. (4.63) as

$$\rho(1)=R(1;2,3)\,A(2)\,\frac{\delta}{\text{i}\,\delta A(3)}\,, \tag{4.88}$$

where

$$R(1;2,3)=-R(1;3,2)=f^{a_1a_2a_3}\,\delta_{i_2i_3}\,\delta(\mathbf{x}_1-\mathbf{x}_2)\,\delta(\mathbf{x}_1-\mathbf{x}_3). \tag{4.89}$$

In this notation, the Coulomb energy Eq. (4.87) reads

$$E_c=\frac{g^2}{8}\,R(1;3,4)\,R(2;5,6)\left\langle F_A(1,2)\,A(3)\,A(5)\frac{\delta S[A]}{\delta A(4)}\,\frac{\delta S[A]}{\delta A(6)}\right\rangle. \tag{4.90}$$

The remaining expectation value can, in principle, be expressed in terms of (so far unknown) higher order vertex functions. The required manipulations are, however, quite involved, and further simplifications are needed for practical reasons. Here we will again restrict ourselves to terms containing up to three overlapping loops in the energy. With this approximation we can factorize the Coulomb kernel $F_A$ (see also Ref. [4]) as

$$E_c \simeq \frac{g^2}{8}\, R(1;3,4)\, R(2;5,6)\, \langle F_A(1,2)\rangle \left\langle A(3)\, A(5)\, \frac{\delta S[A]}{\delta A(4)}\, \frac{\delta S[A]}{\delta A(6)} \right\rangle . \tag{4.91}$$

The Coulomb propagator $\langle F_A\rangle$ is discussed later in Sec. 4.6.3. The remaining expectation value has precisely the form (4.67) with $f[A] = AA$, and from Eq. (4.69) we obtain

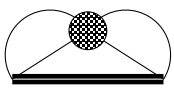
$$\begin{aligned}
&\left\langle \frac{\delta S[A]}{\delta A(4)}\, \frac{\delta S[A]}{\delta A(6)}\, A(3)\, A(5) \right\rangle = \left\langle \frac{\delta^2 S[A]}{\delta A(4)\, \delta A(6)}\, A(3)\, A(5) \right\rangle + \left\langle \frac{\delta^2}{\delta A(4)\, \delta A(6)} [A(3)\, A(5)] \right\rangle \\
&+ \widetilde{\Gamma}_0(4;8,7) \left\langle \frac{\delta}{\delta A(6)} [A(3)\, A(5)]\, G_A(7,8) \right\rangle + \widetilde{\Gamma}_0(6;8,7) \left\langle \frac{\delta}{\delta A(4)} [A(3)\, A(5)]\, G_A(7,8) \right\rangle \\
&+ \widetilde{\Gamma}_0(4;8,7)\, \widetilde{\Gamma}_0(6;8',7')\, \big\langle A(3)\, A(5) \big[ G_A(7,8)\, G_A(7',8') - G_A(7,8')\, G_A(7',8) \big] \big\rangle .
\end{aligned} \tag{4.92}$$

When inserted into Eq. (4.91), this expression can be simplified by exploiting the colour antisymmetry of the vertices $R$, see Eq. (4.89): for this reason, the part of Eq. (4.92) symmetric with respect to the interchange of the indices $(3 \leftrightarrow 4)$ or $(5 \leftrightarrow 6)$ vanishes. Furthermore, the two terms on the r.h.s. of Eq. (4.92) with a single (bare) ghost-gluon vertex $\widetilde{\Gamma}_0$ yield identical contributions when inserted into Eq. (4.91).

The first four terms on the right-hand side of Eq. (4.92) can be straightforwardly evaluated as in the preceding sections. With the explicit form of the 'action' Eq. (4.46), one obtains for these terms

$$\begin{aligned}
&t(3,6)\, t(4,5) + 2\, \widetilde{\Gamma}_0(4;8,7)\, t(3,6)\, \langle A(5)\, G_A(7,8)\rangle + 2\, \omega(4,6)\, \langle A(3) A(5)\rangle + \\
&+ \gamma(4,6,7)\, \langle A(7)\, A(3)\, A(5)\rangle + \tfrac{1}{2}\, \gamma(4,6,7,8)\, \langle A(7)\, A(8)\, A(3)\, A(5)\rangle = \\
&= t(3,6) \big[ t(4,5) - 4\chi(4,7) D(7,5) \big] + 2\omega(4,6) D(3,5) + \\
&- \gamma(4,6,7)\, D(5,5')\, D(3,3')\, D(5,5')\, \Gamma(3',5',7') + \tfrac{1}{2}\, \gamma(4,6,7,8)\, \langle A(7) A(8) A(3) A(5)\rangle ,
\end{aligned} \tag{4.93}$$

where we have used Eqs. (4.21), (4.22), and the definition of the ghost loop [Eq. (4.53b)]. We are still left with the four-point function in Eq. (4.93), and with the six-point function in the last term in Eq. (4.92). To work out these terms, we notice that the Coulomb energy Eq. (4.91) can be diagrammatically represented as



where the "blob" (including the four lines attached to it) represents the v.e.v. given by the last bracket in Eq. (4.91), and the double line stands for the Coulomb propagator $\langle F_A\rangle$. Since we need the energy up to three loops, we should keep only those contributions to the "blob" which either factorize in two disconnected lines or which are either irreducible or at most one-particle reducible (no box diagrams). At this order, for the last term in

Eq. (4.93) it is sufficient to keep from the expression for $\langle A^4 \rangle$ given in Eq. (4.39) only the disconnected terms, yielding

$$\begin{aligned}
&\tfrac{1}{2}\,\gamma(4,6,7,8)\,\langle A(7)A(8)A(3)A(5)\rangle = \\
&= \tfrac{1}{2}\,\gamma(4,6,7,8)\,[D(3,5)\,D(7,8) + D(3,7)\,D(5,8) + D(3,8)\,D(5,7) + \ldots] \\
&= D(3,5)\,2\,\phi_t(4,6) + \gamma(4,6,7,8)\,D(7,3)\,D(8,5) + \ldots
\end{aligned} \tag{4.94}$$

where we have used the definition of gluon tadpole $\phi_t$ (4.53e) and made use of the symmetry properties of the four-gluon kernel $\gamma_4$. The dots in Eq. (4.94) stand for terms which give rise to energy contributions with more than three loops. Following the same line of reasoning, the last term in Eq. (4.92) can be transformed to

$$\begin{aligned}
&\widetilde{\Gamma}_0(4;8,7)\,\widetilde{\Gamma}_0(6;8',7')\,\big\langle A(3)\,A(5)\big[G_A(7,8)\,G_A(7',8') - G_A(7,8')\,G_A(7',8)\big]\big\rangle \\
&= \widetilde{\Gamma}_0(4;8,7)\,\widetilde{\Gamma}_0(6;8',7')\big\{D(3,5)\,\big\langle\big[G_A(7,8)\,G_A(7',8') - G_A(7,8')\,G_A(7',8)\big]\big\rangle \\
&\qquad\qquad + \big\langle A(3)\,G_A(7',8')\big\rangle\,\langle A(5)\,G_A(7,8)\rangle + \ldots\big\} \\
&= D(3,5)\big[-2\,\chi(4,6) + 4\,\chi(4,7)D(7,8)\chi(8,6) + \ldots\big] \\
&\qquad\qquad + 4D(3,7)\chi(7,6)D(5,8)\chi(8,4) + \ldots
\end{aligned} \tag{4.95}$$

where we used Eqs. (4.21), (4.44) and discarded terms involving more than three loops in the energy. Collecting all terms given by Eqs. (4.92)–(4.95) and inserting the result into (4.91), we finally obtain the Coulomb energy to the desired (three-loop) order

$$\begin{aligned}
E_c = \frac{g^2}{8}\,&R(1;3,4)\,R(2;5,6)\,F(1,2)\times \\
&\times\Big\{2\,D(3,5)\big[\omega(4,6) + \phi_t(4,6) - \chi(4,6) + 2\,\chi(4,7)\,D(7,8)\,\chi(8,6)\big] \\
&\quad + t(3,6)\big[t(4,5) - 4\chi(4,7)D(7,5)\big] + 4D(3,7)\chi(7,6)D(5,8)\chi(8,4) \\
&\quad - \gamma(4,6,7)\,D(5,5')\,D(3,3')\,D(5,5')\,\Gamma(3',5',7') \\
&\quad + \gamma(4,6,7,8)\,D(7,3)\,D(8,5)\Big\}.
\end{aligned} \tag{4.96}$$

where $F(1,2) = \langle F_A(1,2)\rangle$. For later use we rewrite this expression in momentum space. Exploiting the symmetries of the entries and expressing the variational kernel $\omega$ through the inverse gluon propagator $\Omega$, as we did for the kinetic energy, Eq. (4.96) can be cast into the form

$$\begin{aligned}
E_c = g^2\,&\frac{N_c(N_c^2-1)}{16}\,V\int \text{đ}p\;\text{đ}q\;F(\mathbf{p}+\mathbf{q})\,\frac{[d-2+(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2]}{\Omega(\mathbf{p})\,\Omega(\mathbf{q})} \\
&\times\Big\{[\Omega(\mathbf{p}) - \chi(\mathbf{p}) - \Omega(\mathbf{q}) + \chi(\mathbf{q})]^2 + \xi(\mathbf{p})\,\Omega(\mathbf{p}) + \xi(\mathbf{q})\,\Omega(\mathbf{q})\Big\} \\
&- V\,\frac{g^2}{8}\int \text{đ}p\;\text{đ}q\;\text{đ}\ell\;F(\boldsymbol{\ell})\frac{t_{im}(\mathbf{p})\,t_{jn}(\mathbf{q})\,t_{kl}(\mathbf{p}+\mathbf{q})}{\Omega(\mathbf{p})\,\Omega(\mathbf{q})\,\Omega(\mathbf{p}+\mathbf{q})}\,f^{gad}f^{gbe} \\
&\qquad\times\gamma^{cde}_{lmn}(\mathbf{p}+\mathbf{q},\boldsymbol{\ell}-\mathbf{p},-\mathbf{q}-\boldsymbol{\ell})\,\Gamma^{cab}_{kij}(-\mathbf{p}-\mathbf{q},\mathbf{p},\mathbf{q}) \\
&+ V\,\frac{g^2}{8}\int \text{đ}p\;\text{đ}q\;\text{đ}\ell\;F(\boldsymbol{\ell})\frac{t_{ij}(\mathbf{p})\,t_{lm}(\mathbf{q})}{\Omega(\mathbf{p})\,\Omega(\mathbf{q})}\,f^{abc}f^{ade}\gamma^{bdce}_{iljm}(\boldsymbol{\ell}-\mathbf{p},\mathbf{p},-\boldsymbol{\ell}-\mathbf{q},\mathbf{q})
\end{aligned} \tag{4.97}$$

where $F(\mathbf{k})$ is the Fourier representation of the Coulomb propagator $\langle F_A\rangle$, see Sec. 4.6.3 below. If we discard the three- and four-gluon kernels $\gamma_3$ and $\gamma_4$, which also removes the

gluon loop $\xi(\mathbf{p})$ [Eq. (4.53a)], this expression reduces to the Coulomb energy obtained in Refs. [4,5] with a Gaussian wave functional,

$$E_c = g^2\,\frac{N_c(N_c^2-1)}{16}\,V\int \text{đ}p\;\text{đ}q\;F(\mathbf{p}+\mathbf{q})\;\frac{[d-2+(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2]}{\Omega(\mathbf{p})\,\Omega(\mathbf{q})}\times \\ \times\left[\Omega(\mathbf{p})-\chi(\mathbf{p})-\Omega(\mathbf{q})+\chi(\mathbf{q})\right]^2\big], \quad (4.98)$$

The new features introduced by the inclusion of the three- and four-gluon kernels will be studied in the subsequent sections.

## 4.6 Determination of the variational kernels

In the previous section we have expressed the vacuum energy $\langle\psi|\,H\,|\psi\rangle$ in terms of the variational kernels $\omega$, $\gamma_3$, and $\gamma_4$ (occurring in our ansatz Eq. (4.10), (4.46) for the vacuum wave functional $\psi[A]$) and of the proper vertices $\Gamma_n$, $\widetilde{\Gamma}_n$. We now use the DSEs (4.52), (4.55), (4.56), and (4.61) to express the proper vertex functions $\Gamma_n$, $\widetilde{\Gamma}_n$ occurring in the energy in terms of the variational kernels $\gamma_n$. We are then in a position to determine these kernels by minimizing $\langle\psi|\,H\,|\psi\rangle$. To make the calculations feasible, we will resort to a skeleton expansion of $\langle\psi|\,H\,|\psi\rangle$, keeping at most three-loop terms. As we will see, this is the minimum number of loops required to obtain a non-trivial four-gluon kernel $\gamma_4$. In the variation of the energy with respect to $\frac{1}{2}\gamma_2=\omega$ and $\gamma_3$ we will restrict ourselves up to two loops terms in $\langle\psi|\,H\,|\psi\rangle$, which will be sufficient to get a non-trivial $\gamma_3$ and a one loop gap equation for $\omega$.

### 4.6.1 Three- and four-gluon kernel

Below we determine the three-gluon kernel in leading order in the number of loops. For this purpose, it is sufficient to keep up to two-loop terms in the energy. The relevant contributions come then from the gluon loop $\xi$ [Eq. (4.79)] occurring in the kinetic energy [Eq. (4.77)], and the magnetic energy contribution (4.83). These terms contain the three-gluon kernel $\gamma_3$ either explicitly or implicitly via the three-point proper vertex $\Gamma_3$, which by its DSE (4.55) is given in lowest order by the three-gluon kernel $\gamma_3$. All remaining terms of $\Gamma_3$ contain additional loops and will henceforth be discarded, resulting in the "tree-level" expression $\Gamma_3=\gamma_3$. Inserting this expression in Eqs. (4.79) and (4.83), and taking into account the symmetry of these kernels, the variation of these energy terms with respect to the three-gluon kernel $\gamma_3$ leads to

$$\frac{\delta}{\delta\gamma_3}\int \text{đ}p\;\text{đ}q\;\text{đ}\ell\,\frac{\delta(\mathbf{p}+\mathbf{q}+\boldsymbol{\ell})}{\Omega(\mathbf{p})\,\Omega(\mathbf{q})\,\Omega(\boldsymbol{\ell})}\left[(\gamma_3\circ\gamma_3)\frac{\Omega(\mathbf{p})+\Omega(\mathbf{q})+\Omega(\boldsymbol{\ell})}{4}-(\gamma_3\circ gT_3)\right]\stackrel{!}{=}0, \quad (4.99)$$

which fixes the three-gluon kernel to

$$\gamma^{abc}_{ijk}(\mathbf{p},\mathbf{q},\mathbf{k})=\frac{2\,g\,T^{abc}_{ijk}(\mathbf{p},\mathbf{q},\mathbf{k})}{\Omega(\mathbf{p})+\Omega(\mathbf{q})+\Omega(\mathbf{k})}\,, \quad (4.100)$$

where the tensor $T^{abc}_{ijk}(\mathbf{p},\mathbf{q},\mathbf{k})$ is defined in Eq. (4.66a). The obtained three-gluon kernel $\gamma_3$ is reminiscent of the perturbative one following from a solution of the Yang–Mills

Schrödinger equation in leading order in the coupling constant $g$ [25]

$$\gamma^{(0)abc}_{ijk}(\mathbf{p},\mathbf{q},\mathbf{k}) = \frac{2\,g\,T^{abc}_{ijk}(\mathbf{p},\mathbf{q},\mathbf{k})}{|\mathbf{p}|+|\mathbf{q}|+|\mathbf{k}|}\,, \tag{4.101}$$

except that the perturbative gluon energy $|\mathbf{k}|$ is replaced in Eq. (4.100) by the non-perturbative one $\Omega(\mathbf{k})$. Note that, in principle, the Lorentz indices in Eqs. (4.100) and (4.101) are contracted with transverse projectors, which we did not explicitly write down, since they arise naturally as these kernels are always contracted with either transverse gauge fields or the corresponding transverse Green functions.

The variational determination of the four-gluon kernel $\gamma_4$ is technically somewhat more involved. The terms of $\langle H\rangle$ contributing to the variation with respect to $\gamma_4$ are those containing $\gamma_4$ either explicitly or implicitly via the DSEs (4.55), (4.56) for the proper three- and four-gluon vertices $\Gamma_3$, $\Gamma_4$. Terms explicitly containing $\gamma_4$ are the $\phi_{1,2}$ terms [Eqs. (4.53c) and (4.53d)] of the kinetic energy [Eq. (4.77)], as well as the last term of the Coulomb energy [Eq. (4.97)]. Terms containing $\Gamma_3$ or $\Gamma_4$ are given by the gluon loop $\xi$ [Eq. (4.79)] in the kinetic energy [Eq. (4.77)] as well as by the magnetic energy contributions (4.83) and (4.85). All terms contributing to the variation of the energy with respect to $\gamma_4$ are collected below:

$$\begin{aligned}
&\frac{1}{4!2}\,\gamma(1,2,3,4)\,D(1,1')\,D(2,2')\,D(3,3')\,\gamma(1',2',3',4)\\
&-\frac{1}{16}\,\gamma(1,2,3,4)\,\gamma(3',4',5)\,D(5,5')\,\gamma(5',1,2')\,D(2,2')\,D(3,3')\,D(4,4')+\\
&+\Gamma(1,2,3)\,D(1,1')\,D(2,2')\left[\frac{1}{16}\,\gamma(1',2',3)-\frac{1}{3!}\,D(3,3')\,T(1',2',3')\right]\\
&-\frac{1}{4!}\,\gamma(1,2,3,4)\,D(1,1')\,D(2,2')\,D(3,3')\,D(4,4')\,\gamma(1',2',3',4')\\
&+\frac{g^2}{8}\,F(1,2)\,R(1;3,4)\,R(2;5,6)\,\gamma(4,6,7,8)\,D(7,3)\,D(8,5)\,.
\end{aligned} \tag{4.102}$$

For simplicity, we have not explicitly symmetrized this expression with respect to a permutation of the indices of $\gamma_4$.[4] In the above expression we have used the DSE (4.56) to replace the four-gluon vertex $\Gamma_4$ by the four-gluon kernel $\gamma_4$. To be consistent, we have to use the DSE for the three-gluon vertex [Eq. (4.55)], to express the vertex function $\Gamma_3$ in Eq. (4.102) by the variational kernels, where it is sufficient to retain only terms involving $\gamma_4$ or $\Gamma_4$, and the latter is to be replaced by $\gamma_4$ due to the DSE (4.56). Taking into account the symmetry properties of the quantities involved, by Eq. (4.55) we are led to make the following replacement in Eq. (4.102)

$$\begin{aligned}
\Gamma(1,2,3)\to\ &-\frac{1}{2}\,\gamma(3,4,5)\,D(4,4')\,D(5,5')\,\gamma(4',5',1,2)\\
&-\gamma(3,1,4,5)\,D(4,4')\,D(5,5')\,\gamma(4',5',2)\,.
\end{aligned} \tag{4.103}$$

[4] Recall that, by definition, $\gamma_4$ is totally symmetric with respect to a permutation of indices. Strictly speaking, one should first symmetrize Eq. (4.102) and then take the variation. However, the same result is more conveniently obtained by taking the variation of the unsymmetrized expression and symmetrizing afterwards.

With this replacement, the variation of Eq. (4.102) with respect to $\gamma_4$ is now straightforward and yields after proper symmetrization with respect to external indices of $\gamma_4$

$$\begin{aligned}
&\big[\Omega(\mathbf{k}_1)+\Omega(\mathbf{k}_2)+\Omega(\mathbf{k}_3)+\Omega(\mathbf{k}_4)\big]\,\gamma^{abcd}_{ijkl}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = 2\,g^2\,T^{abcd}_{ijkl}\\
&\quad -\frac{1}{2}\Big\{\gamma^{abe}_{ijm}(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}_1-\mathbf{k}_2)\,t_{mn}(\mathbf{k}_1+\mathbf{k}_2)\,\gamma^{cde}_{kln}(\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_1+\mathbf{k}_2)\\
&\qquad +\gamma^{ace}_{ikm}(\mathbf{k}_1,\mathbf{k}_3,-\mathbf{k}_1-\mathbf{k}_3)\,t_{mn}(\mathbf{k}_1+\mathbf{k}_3)\,\gamma^{bde}_{jln}(\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_1+\mathbf{k}_3)\\
&\qquad +\gamma^{ade}_{ilm}(\mathbf{k}_1,\mathbf{k}_4,-\mathbf{k}_1-\mathbf{k}_4)\,t_{mn}(\mathbf{k}_1+\mathbf{k}_4)\gamma^{bce}_{jkn}(\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_1+\mathbf{k}_4)\Big\}\\
&-2g^2\Big\{f^{abe}f^{cde}\delta_{ij}\delta_{kl}\big[\Omega(\mathbf{k}_1)-\Omega(\mathbf{k}_2)\big]F(\mathbf{k}_1+\mathbf{k}_2)\big[\Omega(\mathbf{k}_3)-\Omega(\mathbf{k}_4)\big]\\
&\qquad +f^{ace}f^{bde}\delta_{ik}\delta_{jl}\big[\Omega(\mathbf{k}_1)-\Omega(\mathbf{k}_3)\big]F(\mathbf{k}_1+\mathbf{k}_3)\big[\Omega(\mathbf{k}_2)-\Omega(\mathbf{k}_4)\big]\\
&\qquad +f^{ade}f^{bce}\delta_{il}\delta_{jk}\big[\Omega(\mathbf{k}_1)-\Omega(\mathbf{k}_4)\big]F(\mathbf{k}_1+\mathbf{k}_4)\big[\Omega(\mathbf{k}_2)-\Omega(\mathbf{k}_3)\big]\Big\}.
\end{aligned}\tag{4.104}$$

$F(\mathbf{p})$ is again the Fourier representation of the Coulomb propagator $\langle F_A\rangle$. Equation (4.104) yields a four-gluon kernel $\gamma_4$, which is reminiscent of the perturbative one [25] except that the perturbative propagators and vertices are replaced by the full ones.

### 4.6.2 Gap equation

Given the explicit form of the energy functional, Eqs. (4.77), (4.83)–(4.86), (4.97), it is more convenient to use the DSE (4.52) for the gluon propagator to express the kernel $\omega(\mathbf{p})$ in terms of the gluon energy $\Omega(\mathbf{p})$, and vary the energy density with respect to $\Omega^{-1}(\mathbf{p})$. The vacuum energy is given by closed loop diagrams, and the variation with respect to the gluon propagator $\Omega^{-1}(\mathbf{p})$ reduces the number of loops by one. If the (gap) equation for $\Omega(\mathbf{p})$ is to be calculated up to one loop, it is sufficient to keep up to two loops in the energy. At this order, the energy density $\varepsilon$ defined by $\langle H\rangle =: V(d-1)(N_c^2-1)\varepsilon$ is given by

$$\begin{aligned}
\varepsilon &= \frac{1}{4}\int \text{đ}p\,\frac{\mathbf{p}^2+[\Omega(\mathbf{p})-\chi(\mathbf{p})]^2}{\Omega(\mathbf{p})}\\
&\quad -\frac{g^2\,N_\text{c}}{3!8(d-1)}\int \text{đ}p\,\text{đ}q\frac{\gamma_3}{\Omega(\mathbf{p})\Omega(\mathbf{q})\Omega(\mathbf{p}+\mathbf{q})}\circ\left[\frac{\Omega(\mathbf{p})+\Omega(\mathbf{q})+\Omega(\mathbf{p}+\mathbf{q})}{4}\,\gamma_3-gT_3\right]\\
&\quad +\frac{g^2\,N_\text{c}}{16(d-1)}\int \text{đ}p\;\text{đ}q\,[d-2+(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2]F(\mathbf{p}+\mathbf{q})\frac{[\Omega(\mathbf{p})-\chi(\mathbf{p})-\Omega(\mathbf{q})+\chi(\mathbf{q})]^2}{\Omega(\mathbf{p})\,\Omega(\mathbf{q})}\,,
\end{aligned}\tag{4.105}$$

see Eqs. (4.77), (4.83) and (4.98). In Eq. (4.105) we have discarded the tadpole term, Eq. (4.84) or (4.84′), since it represents an irrelevant constant, which disappears after renormalization. Except for the gluon loop (second term), the energy density Eq. (4.105) was already obtained in Refs. [4,5], where a Gaussian wave functional multiplied by $\mathcal{J}_A^{-1/2}$ was used. The gluon loop is lost when a Gaussian wave functional is used, for which Green's functions with an odd number of fields vanish.

In principle, in Eq. (4.105) the three-gluon kernel $\gamma_3$ [Eq. (4.100)] obtained from the variational principle $\delta\varepsilon/\delta\gamma_3 = 0$ depends on $\Omega(\mathbf{p})$. However, since the energy density $\varepsilon$

Eq. (4.105) with $\gamma_3$ given by Eq. (4.100) is already stationary with respect to variations of $\gamma_3$, we can ignore the implicit $\Omega(\mathbf{p})$ dependence of $\varepsilon$ via $\gamma_3$. Variation of $\varepsilon$ [Eq. (4.105)] with respect to $\Omega^{-1}(\mathbf{k})$ then yields the gap equation

$$\Omega(\mathbf{p})^2 = \mathbf{p}^2 + \chi(\mathbf{p})^2 - I_{\mathrm{G}}(\mathbf{p}) + I_{\mathrm{C}}(\mathbf{p}), \tag{4.106}$$

where $\chi(\mathbf{p})$ is the ghost loop [Eq. (4.78)],

$$I_{\mathrm{G}}(\mathbf{p}) = \frac{1}{4(d-1)(N_{\mathrm{c}}^2-1)} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \, \frac{1}{\Omega(\mathbf{q})\,\Omega(\mathbf{p}+\mathbf{q})} \, \gamma_3 \circ \left[ g T_3 - \gamma_3 \, \frac{\Omega(\mathbf{p}) + \Omega(\mathbf{p}+\mathbf{q})}{4} \right] \tag{4.107a}$$

is the gluon-loop contribution, and

$$\begin{aligned} I_{\mathrm{C}}(\mathbf{p}) = \frac{g^2 \, N_{\mathrm{c}}}{2(d-1)} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \, \left[ d - 2 + (\hat{\mathbf{p}} \cdot \hat{\mathbf{q}})^2 \right] F(\mathbf{p}+\mathbf{q}) \\ \times \, \frac{\left[ \Omega(\mathbf{q}) - \chi(\mathbf{q}) + \chi(\mathbf{p}) \right]^2 - \Omega(\mathbf{p})^2}{\Omega(\mathbf{q})} \end{aligned} \tag{4.107b}$$

arises from the Coulomb term. Except for the gluon loop $I_{\mathrm{G}}(\mathbf{k})$, Eq. (4.106) is the gap equation found already in [4].

If we insert the expression derived above in leading order for the three-gluon kernel $\gamma_3$ [Eq. (4.100)] into the gluon-loop contribution [Eq. (4.107a)], this takes the form

$$I_{\mathrm{G}}(\mathbf{p}) = \frac{g^2 \, N_{\mathrm{c}}}{d-1} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \, \frac{2\Omega(\mathbf{p}) + \Omega(\mathbf{q}) + \Omega(\mathbf{p}+\mathbf{q})}{\left[ \Omega(\mathbf{p}) + \Omega(\mathbf{q}) + \Omega(\mathbf{p}+\mathbf{q}) \right]^2} \, \frac{\Sigma(\mathbf{p},\mathbf{q})}{\Omega(\mathbf{q})\,\Omega(\mathbf{p}+\mathbf{q})} \, , \tag{4.108}$$

where the function $\Sigma$ arises from the contraction of the Lorentz structure of two three-gluon vertices upon imposing momentum conservation

$$T_3 \circ T_3 \big|_{\mathbf{k}=-\mathbf{p}-\mathbf{q}} =: 4 \, N_{\mathrm{c}} \, (N_{\mathrm{c}}^2 - 1) \, \Sigma(\mathbf{p},\mathbf{q}), \tag{4.109}$$

and is given explicitly by

$$\Sigma(\mathbf{p},\mathbf{q}) = \left[ 1 - (\hat{\mathbf{p}} \cdot \hat{\mathbf{q}})^2 \right] \left[ (d-1)(\mathbf{p}^2 + \mathbf{q}^2) + \frac{(d-2)\mathbf{p}^2\mathbf{q}^2 + (\mathbf{p}\cdot\mathbf{q})^2}{(\mathbf{p}+\mathbf{q})^2} \right] . \tag{4.110}$$

To exhibit the UV-behaviour of the various loop terms, let us consider them in leading order in perturbation theory [7,25], where $F(\mathbf{p}) = 1/\mathbf{p}^2$ and $\chi(\mathbf{p}) = 0$. In this order we find with 3-momentum cut-off $\Lambda$ for the divergent parts

$$I_{\mathrm{G}}(\mathbf{p}) = \frac{g^2 N_{\mathrm{c}}}{(4\pi)^2} \left[ \frac{4}{3} \, \Lambda^2 + \frac{22}{15} \, \mathbf{p}^2 \ln \frac{\Lambda^2}{\mathbf{p}^2} \right] , \tag{4.111a}$$

$$I_{\mathrm{C}}(\mathbf{p}) = \frac{g^2 N_{\mathrm{c}}}{(4\pi)^2} \left[ \frac{4}{3} \, \Lambda^2 - \frac{8}{15} \, \mathbf{p}^2 \ln \frac{\Lambda^2}{\mathbf{p}^2} \right] . \tag{4.111b}$$

One observes that the quadratic divergence in Eq. (4.106) cancels, and the sum of the logarithmic ones is consistent with the result of Lagrangian-based perturbation theory in Coulomb gauge [24,28]

$$\frac{|\mathbf{p}|}{\Omega(\mathbf{p})} = 1 + \frac{1}{2\mathbf{p}^2} [I_{\mathrm{G}}(\mathbf{p}) - I_{\mathrm{C}}(\mathbf{p})]) + \ldots = 1 + \frac{g^2 N_{\mathrm{c}}}{(4\pi)^{2-\varepsilon}} \left( \frac{1}{\varepsilon} - \ln \frac{\mathbf{p}^2}{\mu^2} \right) + \ldots \tag{4.112}$$

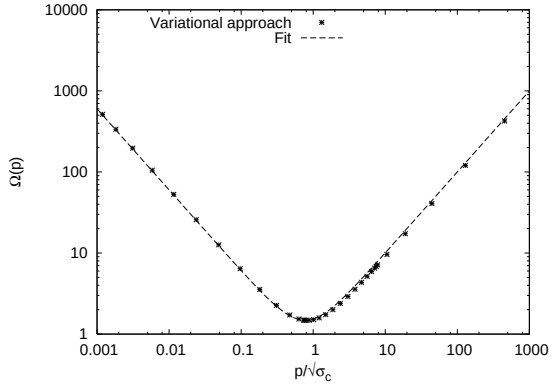



Figure 4.16: Numerical results from Ref. [8] for the gluon energy $\Omega(\mathbf{p})$ and fit to Eq. (4.107a).

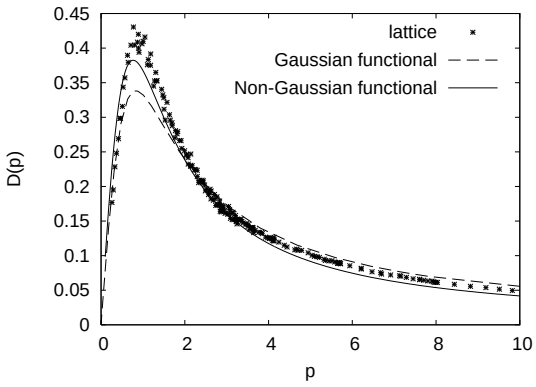



Figure 4.17: Comparison of the gluon propagator obtained in Ref. [8] with a Gaussian wave functional (dashed line) with the corrected one from Eq. (4.114) (full line), and the lattice data from Ref. [9].

where we have used dimensional regularization with $d = 3 - 2\varepsilon$.

To estimate the size of the contribution of the gluon loop [Eq. (4.107a)] to the gap equation (4.106), we use the gluon energy $\Omega$ obtained with a Gaussian wave functional in Ref. [8] as input. For practical purposes, we fit the $\Omega(\mathbf{p})$ obtained in Ref. [8] to the formula

$$\Omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + \frac{m_{\mathrm{A}}^4}{\mathbf{p}^2} + c^2}\,, \tag{4.113}$$

which yields $m_{\mathrm{A}}^4 \simeq 0.36\,\sigma_{\mathrm{C}}^2$, $c^2 \simeq 1.0\,\sigma_{\mathrm{C}}$. The numerical results of Ref. [8] and the fit to Eq. (4.113) are shown in Fig. 4.16. The gluon loop [Eq. (4.107a)] is UV divergent. In principle, the UV-divergent part is removed by the renormalization of the gap equation (4.106), which can be done analogously to Refs. [12,29]. We therefore calculate

$$\Omega_{\mathrm{new}}(\mathbf{p})^2 = \Omega_{\mathrm{old}}(\mathbf{p})^2 - \big(I_{\mathrm{G}}(\mathbf{p})[\Omega_{\mathrm{old}}] - I_{\mathrm{G}}(\mathbf{p})_{\mathrm{div}}\big), \tag{4.114}$$

where $I_{\mathrm{G}}(\mathbf{p})_{\mathrm{div}}$ is the (known) perturbative divergent part of the gluon loop, which has been subtracted, and Eq. (4.113) has been used for $\Omega_{\mathrm{old}}(\mathbf{p})$. The (inverse) gluon propagator $\Omega_{\mathrm{new}}(\mathbf{p})$ Eq. (4.114) is shown in Fig. 4.17 together with the one obtained previously [8] with a Gaussian wave functional $\Omega_{\mathrm{old}}(\mathbf{p})$. The mismatch in the UV is due to the anomalous dimension developed by $\Omega_{\mathrm{new}}(\mathbf{p})$ and absent in $\Omega_{\mathrm{old}}(\mathbf{p})$. As seen in Fig. 4.17, significant correction from the gluon loop arises in the mid-momentum regime, a behaviour which was observed also in Landau gauge [30,31]. This a posteriori supports the use of the Gaussian wave functional for the description of the infrared regime.

### 4.6.3 The Coulomb form factor

With the explicit expression of the three- and four-gluon kernels, $\gamma_3$ [Eq. (4.100)] and $\gamma_4$ [Eq. (4.104)], on hand, we are left with three coupled equations: Eq. (4.59) for the ghost propagator, Eq. (4.61) for the ghost-gluon vertex, and the gap equation (4.106) for the gluon propagator. The final piece which is missing for closing this set of equations is the

non-abelian colour Coulomb potential $F(1,2)$ defined by the vacuum expectation value of the Coulomb kernel [Eq. (4.64)]

$$F(1,4) = \langle F_A(1,4)\rangle = \left\langle G_A(1,2)\, G_0^{-1}(2,3)\, G_A(3,4)\right\rangle . \tag{4.115}$$

The quantity $g^2F(1,2)$ directly relates to the heavy quark potential [32] and is hence a renormalization group invariant quantity. Before we come to the evaluation of $\langle F_A\rangle$, let us remark that in practical application it is not necessary to explicitly solve the DSE (4.61) for the ghost-gluon vertex. Rather it is sufficient to replace the full ghost-gluon vertex $\widetilde{\Gamma}$ [Eq. (4.35)] by the bare one $\widetilde{\Gamma}_0$ [Eq. (4.16)]. This approximation is motivated by the 'non-renormalization' theorem for this vertex [33]. Although this theorem was originally proven [33] and confirmed on the lattice [34,35] for QCD in Landau gauge, the arguments carry over to the present case of Coulomb gauge. A perturbative evaluation of the ghost-gluon vertex in Coulomb gauge shows indeed that its quantum corrections are finite and independent of the scale [7,25].

The vacuum expectation value $\langle F_A\rangle$ is commonly expressed in terms of the Coulomb form factor $f$. This quantity measures the deviation of $\langle F_A\rangle$ from the factorized form $\langle G_A\rangle\, G_0^{-1}\, \langle G_A\rangle$ and is defined in momentum space by

$$F(\mathbf{p}) =: G(\mathbf{p})\, f(\mathbf{p})\, \mathbf{p}^2\, G(\mathbf{p}), \tag{4.116}$$

where $G = \langle G_A\rangle$ is the ghost propagator [Eq. (4.23)]. By taking the vacuum expectation value of the operator identity

$$F_A = \frac{\partial}{\partial g}\big(g\, G_A\big), \tag{4.117}$$

the Coulomb form factor $f(\mathbf{p})$ can be related to the ghost form factor $d(\mathbf{p})$ and from the ghost DSE (4.59) the following (approximate) integral equation for the Coulomb form factor is obtained [4,36]

$$f(\mathbf{p}) = 1 + g^2\frac{N_c}{2}\int \text{đ}q\, \frac{1-(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2}{\Omega(\mathbf{q})}\,(\mathbf{p}-\mathbf{q})^2\, G^2(\mathbf{p}-\mathbf{q})\, f(\mathbf{p}-\mathbf{q}). \tag{4.118}$$

In the derivation of this equation no assumption on the form of the vacuum wave functional enters, so this equation remains also valid for non-Gaussian wave functionals.

With the approximation $\widetilde{\Gamma}$ [Eq. (4.35)] $\to$ $\widetilde{\Gamma}_0$ [Eq. (4.16)], Eqs. (4.59), (4.106), and (4.118) form a closed set of coupled equations for the ghost propagator $G(\mathbf{p})$, the gluon energy $\Omega(\mathbf{p})$, and the Coulomb form factor $f(\mathbf{p})$, whose solutions provide the variational solution of the Yang–Mills Schrödinger equation. These equations have to be renormalized, which can be done in exactly the same way as in Refs. [4,12,29]. The numerical solution of this set of equations can be carried out as in the case of the Gaussian wave functional [4,8] and is subject to future work.

As already mentioned before, the use of a Gaussian wave functional misses the contribution of the bare three-gluon vertex and thus the gluon loop in the gap equation. Fortunately, as we have seen in the previous subsection, the gluon loop is IR subleading compared to the (included) ghost loop, and thus irrelevant for the IR behaviour of the Green functions. Therefore the results of Refs. [4,8] remain fully valid in the IR. In particular, the gluon propagator does not change in the IR. The gluon loop does, however, matter in the UV and is responsible for the anomalous dimension of the gluon propagator, which enters the running coupling constant.

## 4.7 The three- and four-gluon vertex

As we have seen, the gap equation differs from the one obtained in [4] only by the gluon loop contribution, which gives sizeable corrections in the mid-momentum regime. In the present section, we investigate the three- and four-gluon vertices $\Gamma_{3,4}$ by using the ghost and gluon propagators determined with a Gaussian wave functional [8] as input. We will not resort to the tree-level result $\Gamma_n = \gamma_n$ but rather solve the corresponding DSEs to one-loop order.

### 4.7.1 Solution of the truncated three-gluon vertex DSE

The DSE for the three gluon-vertex at one-loop order is given by Eq. (4.55). Assuming ghost dominance (see for example Ref. [6]), we keep only the ghost-loop (the third term on the right-hand side of Eq. (4.55) and Fig. 4.9). Furthermore, we replace the full ghost-gluon vertex by the bare one, see Sec. 4.6.3. After extracting the colour structure, the truncated DSE for the three-gluon vertex becomes

$$\Gamma_{ijk}(\mathbf{p},\mathbf{q},\mathbf{k}) = \gamma_{ijk}(\mathbf{p},\mathbf{q},\mathbf{k}) + \mathrm{i}g^3 N_\mathrm{c} \int \text{đ}\ell\, G(\boldsymbol{\ell})\, G(\boldsymbol{\ell}-\mathbf{p})\, G(\boldsymbol{\ell}+\mathbf{q}) \ell_i \ell_j (\ell - p)_k \,, \quad (4.119)$$

where momentum conservation $\mathbf{k}+\mathbf{p}+\mathbf{q}=0$ is implied and we have omitted longitudinal terms, which, by definition, cannot enter the proper $n$-point vertex functions Eq. (4.36) of the transverse (Coulomb) gauge field. Possible tensor decompositions of the three-gluon proper vertex are given in Refs. [37,38]. Here, for sake of illustration, we confine ourselves to the form factor corresponding to the tensor structure of the bare three-gluon vertex, defined by

$$f_{3A} := \frac{\Gamma_3^{(0)} \circ \Gamma_3}{\Gamma_3^{(0)} \circ \Gamma_3^{(0)}} \,, \quad (4.120)$$

where $\Gamma_3^{(0)}$ is the perturbative vertex given in Eq. (4.101). Restricting the kinematic configuration to the case where two external momenta have the same magnitude, i.e.

$$\mathbf{q}^2 = \mathbf{p}^2 = p^2, \qquad \mathbf{q}\cdot\mathbf{p} = cp^2, \quad (4.121)$$

the form factor Eq. (4.120) depends only on two variables, the magnitude $p$ of the two momenta and the cosine $c$ of the angle between them. Contracting Eq. (4.119) with the bare three-gluon vertex Eq. (4.101), thereby using the variational kernel $\gamma_3$ [Eq. (4.100)] and the tensor structure $T_3$ given in Eq. (4.66a), yields the following equation for the form factor $f_{3A}$ [Eq. (4.120)]

$$\begin{aligned} f_{3A}(p^2,c) = {} & \frac{2p + \sqrt{2p^2(1+c)}}{2\Omega(p^2) + \Omega(2p^2(1+c))} \\ & - g^2 \frac{N_\mathrm{c}}{4} \frac{2+\sqrt{2(1+c)}}{p(1-c)(9+8c+c^2)} \int \text{đ}\ell\, G(\boldsymbol{\ell})\, G(\boldsymbol{\ell}-\mathbf{p})\, G(\boldsymbol{\ell}+\mathbf{q})\, \mathcal{B}(\boldsymbol{\ell},\mathbf{p},\mathbf{q}), \end{aligned} \quad (4.122)$$

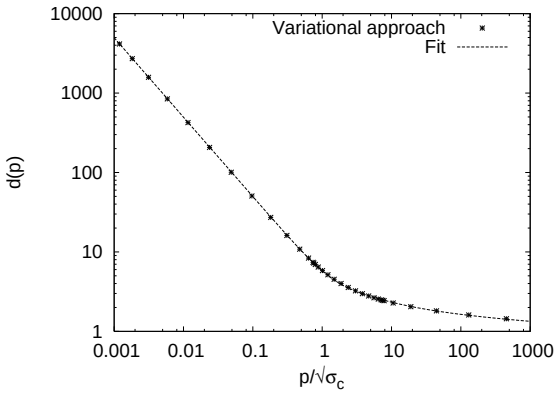



Figure 4.18: Ghost form factor from [8] and fit to Eq. (4.124).

where we have introduced the abbreviation

$$\begin{aligned}\mathcal{B}(\boldsymbol{\ell},\mathbf{p},\mathbf{q}) = \boldsymbol{\ell}^2\big[p^2(1-c)-(1+2c)(\boldsymbol{\ell}\cdot\mathbf{q}-\boldsymbol{\ell}\cdot\mathbf{p})\big] + \frac{(\boldsymbol{\ell}\cdot\mathbf{q})^3}{p^2} - \frac{(\boldsymbol{\ell}\cdot\mathbf{p})^3}{p^2} \\ - (1+c)\big[(\boldsymbol{\ell}\cdot\mathbf{q})^2+(\boldsymbol{\ell}\cdot\mathbf{p})^2\big] \\ + (\boldsymbol{\ell}\cdot\mathbf{q})(\boldsymbol{\ell}\cdot\mathbf{p})\big[c(1+c)+2+\frac{1-c}{p^2}(\boldsymbol{\ell}\cdot\mathbf{q}-\boldsymbol{\ell}\cdot\mathbf{p})\big].\end{aligned} \tag{4.123}$$

The apparent singularity at $c = 1$ in the coefficient in front of the integral in Eq. (4.122) is cancelled by the numerator, and the whole term is regular in the collinear limit. Also the singularities at $\boldsymbol{\ell} = 0$, $\boldsymbol{\ell} = \mathbf{p}$, and $\boldsymbol{\ell} = -\mathbf{q}$ are integrable.

For the actual calculation of $f_{3A}$ [Eq. (4.120)] we use the numerical result for $\Omega(\mathbf{p})$ and $G(\mathbf{p})$ obtained in Ref. [8] (with a Gaussian wave functional) as input. For $\Omega(\mathbf{p})$ we use the parametrization Eq. (4.113) while the ghost form factor $d(\mathbf{p}) = \mathbf{p}^2 G(\mathbf{p})$ is parametrized as

$$d(x) = a\sqrt{\frac{1}{x^2}+\frac{1}{\ln(x^2+c^2)}}\,, \qquad x^2 = \frac{\mathbf{p}^2}{\sigma_{\mathrm{C}}}\,, \tag{4.124}$$

with $\sigma_{\mathrm{C}}$ being the Coulomb string tension. The fit to the data shown in Fig. 4.18 yields $a \simeq 5$ and $c \simeq 4$.

The form Eq. (4.124) of the ghost form factor embodies also the non-perturbative anomalous dimension, which in turn guarantees the convergence of the integral in Eq. (4.122).

The numerical results for $f_{3A}(p^2, c)$ are shown in Fig. 4.19 for some values of the cosine of the relative angle, $c$. The logarithmic plot (left panel) shows that the curves for different $c$ have the same power law behaviour in the infrared region, namely $p^{-3}$, in agreement with the IR analysis of the ghost loop carried out in Ref. [38], according to which the IR exponent of the form factor $f_{3A}$ should be three times the one of the ghost dressing function Eq. (4.124).[5] The linear plot (right panel) shows that the form factor approaches unity in the high-momentum regime and changes sign in the mid momentum regime.

[5] In Ref. [6], due to the use of a Gaussian wave functional, the bare term escaped, and only the ghost loop was calculated for a different solution of the DSEs [4], which leads to less IR singular ghost and gluon propagators.

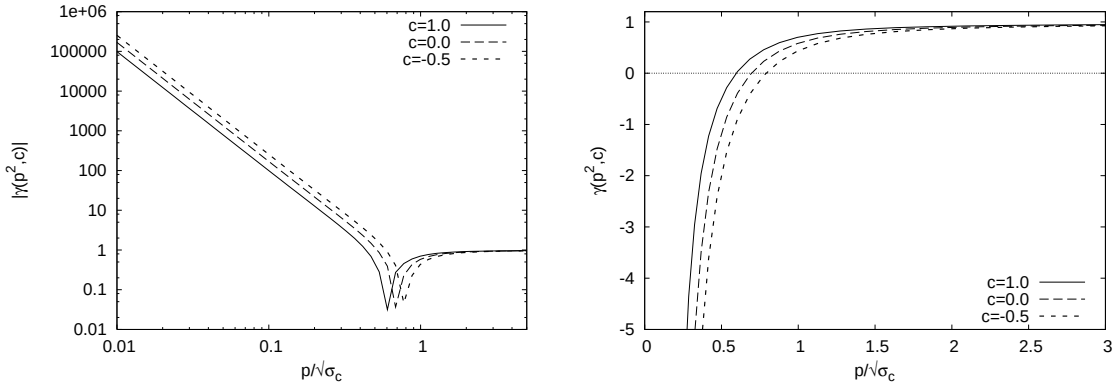



Figure 4.19: Form factor $f_{3A}(p^2, c)$ defined in Eq. (4.120) for two momenta of equal magnitude $p$ plotted for various values of the relative angle $c$.

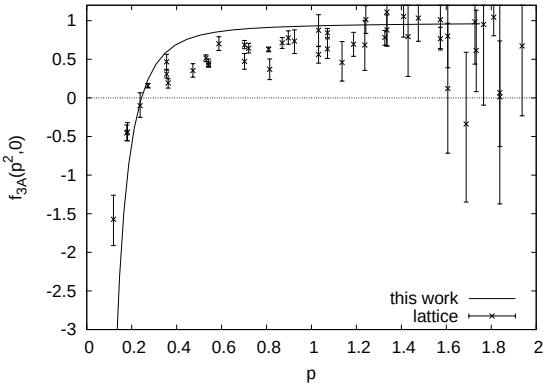



Figure 4.20: Form factor $f_{3A}$ of the three-gluon vertex for orthogonal momenta and comparison to lattice data for the $d = 3$ Landau-gauge vertex [39]. The momentum scale is arbitrary and has been adjusted to make the sign change occur at the same point. The lattice data are shown by courtesy of A. Maas.

In Ref. [39] the form factor $f_{3A}$ [Eq. (4.120)] of the three-gluon vertex was evaluated on the lattice for $d = 3$ Yang–Mills theory in Landau gauge. The result is shown in Fig. 4.20 and compares well to our result in low-momentum regime. In particular, in both studies, the sign change of the form factor occurs roughly at the same momentum where the gluon propagator has its maximum. The $d = 3$ Yang–Mills theory in Landau gauge can be interpreted as using the wave functional

$$\psi[A] = \mathcal{N} \exp\left[-\frac{1}{4\mu^2}\int (F^a_{ij})^2\right] \tag{4.125}$$

in the Hamiltonian approach to $d = 3+1$ Yang–Mills theory in Coulomb gauge. This wave functional can be considered to represent the strong coupling limit of the true vacuum wave functional [40]. In Ref. [41] it was shown by means of lattice calculations that this wave functional yields static propagators which in the IR compare well with those obtained in $d = 3+1$ Yang–Mills theory in Coulomb gauge. Thus we can conclude from Fig. 4.20 that our results compare favourably with the lattice data.

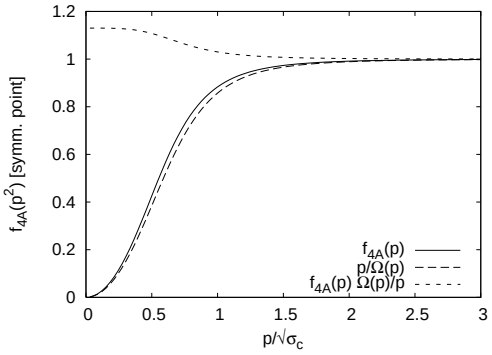



Figure 4.21: Form factor $f_{4A}(p^2)$ of the four-gluon vertex at the symmetric point. The leading contribution $p/\Omega(p)$ is also shown.

### 4.7.2 Estimate of the four-gluon vertex

Because of its DSE (4.56), the four-gluon vertex is given in leading order (neglecting loops) by the variational kernel $\gamma_4$ [Eq. (4.104)] determined in Sec. 4.6.1. A form factor for the four-gluon vertex can be introduced along the same line of Eq. (4.120)

$$f_{4A} := \frac{\Gamma_4^{(0)} \circ \gamma_4}{\Gamma_4^{(0)} \circ \Gamma_4^{(0)}}\,, \tag{4.126}$$

where $\Gamma_4^{(0)}$ is the perturbative four-gluon vertex, which is obtained from Eq. (4.104) by the replacements $\Omega(\mathbf{p}) \to |\mathbf{p}|$, $F(\mathbf{p}) \to 1/\mathbf{p}^2$. We consider here the form factor $f_{4A}$ at the symmetric point, where

$$\mathbf{p}_1^2 = \ldots = \mathbf{p}_4^2 = p^2, \qquad \mathbf{p}_i \cdot \mathbf{p}_j = -\frac{1}{3}\,p^2 \quad (i \neq j). \tag{4.127}$$

For this kinematic configuration the terms in Eq. (4.104) stemming from the Coulomb interaction do not contribute to this vertex, and one finds for the form factor Eq. (4.126)

$$f_{4A}(p^2) = \frac{p}{\Omega(p^2)}\,\frac{171 - 960[g_0 + g(p^2)] + 8704\,g_0\,g(p^2)}{171 - 1920\,g_0 + 8704\,g_0^2}\,, \tag{4.128}$$

where

$$g(p^2) = \left[\frac{p}{2\Omega(p^2) + \Omega(\frac{4}{3}p^2)}\right]^2, \tag{4.129a}$$

$$g_0 = g(p^2)\Big|_{\Omega(p^2)=p} = \tfrac{3}{8}(2 - \sqrt{3}). \tag{4.129b}$$

The function $f_{4A}$ [Eq. (4.128)] is shown in Fig. 4.21. The function multiplying $p/\Omega(p^2)$ in Eq. (4.128) is of $\mathcal{O}(1)$, and the form factor at the symmetric point Eq. (4.127) can be fairly well represented by the gluonic dressing function $p/\Omega(p^2)$ alone.

In the above calculation of the four-gluon vertex we have neglected all loop diagrams, in particular the ghost loop. From analogous investigations in Landau gauge [42] one may expect that the ghost loop dominates the IR behaviour of the four-gluon vertex as it did for the three-gluon vertex. We will defer this issue to future research.

## 4.8 Summary and conclusions

We have presented a general method to treat non-Gaussian wave functionals in the Hamiltonian formulation of quantum field theory. By means of well-established Dyson–Schwinger equation techniques, the equal-time Green functions and, in particular, the expectation value of the Hamiltonian are expressed in terms of kernels occurring in the exponent of the vacuum wave functional. These kernels are then determined by the variational principle minimizing the vacuum energy density. The method was applied to Yang–Mills theory in Coulomb gauge using a vacuum wave functional which contains up to quartic terms in the exponent. In leading order (in the number of loops) the cubic and quartic interaction kernels obtained from the variational principle are reminiscent of the corresponding expressions obtained in leading order of a perturbative solution of the Yang–Mills Schrödinger equation, except that the unperturbed gluon propagators are replaced by the full ones. We have estimated these interaction kernels $\gamma_{3,4}$ and the corresponding proper gluon vertex functions $\Gamma_{3,4}$ by using the gluon and ghost propagators found in the variational approach with a Gaussian wave functional. The resulting three-gluon vertex compares fairly well to the available lattice data obtained in $d = 3$ Landau gauge. The (gap) equation of motion obtained from the variation of the energy density with respect to the gluon propagator contains the gluon loop, which was missed in previous variational approaches due to the use of a Gaussian wave functional. We have shown that the gluon loop gives a substantial contribution in the mid-momentum regime while leaving the IR sector unchanged. Furthermore (together with the contribution from the non-abelian Coulomb interaction already fully included in previous studies [4]), it also provides the correct asymptotic UV behaviour of the gluon propagator in accord with perturbation theory [7]. The approach developed in the present paper allows a systematic treatment of correlations in the Hamiltonian approach to interacting quantum field theories (analogous to the so-called 'exponential $S$' method in many-body physics) and opens up a wide range of applications. In particular, it allows us to extend the variational approach from pure Yang–Mills theory to full QCD. This will be subject to future research.

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Nullus est liber tam malus,
ut non aliqua parte prosit.

*(Gaius Plinius Caecilius Secundus)*

# Article 5

# Topological susceptibility in $SU(2)$ Yang–Mills theory in the Hamiltonian approach in Coulomb gauge

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**Abstract** The topological susceptibility is calculated within the Hamiltonian approach to Yang–Mills theory in Coulomb gauge, using the vacuum wave functional previously determined by a variational solution of the Yang–Mills Schrödinger equation. The numerical result agrees qualitatively with the predictions of lattice simulations.

## 5.1 Introduction

At the classical level, quantum chromodynamics (QCD) with massless fermions is chirally symmetric; that is to say, the QCD Lagrangian is invariant under separate global flavor rotations of the left- and right-handed quarks. For $N_f$ massless quark flavors the chiral symmetry group is

$$SU_V(N_f) \times SU_A(N_f) \times U_V(1) \times U_A(1), \tag{5.1}$$

where the vector (axial vector) symmetry groups denoted by a subscript $V$ ($A$) rotate left- and right-handed fermions in the same (opposite) way. The chiral symmetry is a good starting point for the $N_f = 3$ light quark flavors $u$, $d$, $s$. In the quantum theory, the $SU_A(N_f)$ is spontaneously broken by the dynamical condensation of quarks, $\langle \bar{q}q \rangle \neq 0$, which results in the generation of a constituent quark mass and gives rise to $N_f^2 - 1$ Goldstone bosons, which can be identified with the octet of light pseudoscalar mesons. Further, the small but finite (current) quark masses break the axial $SU_A(N_f)$ explicitly and induce a mass for the pseudoscalar mesons. Furthermore, the $SU_V(N_f)$ is softly broken by the differences in the current quark masses, which lifts the mass degeneracy of the pseudoscalar mesons. The $U_V(1)$ symmetry corresponds to baryon number conservation and remains unbroken in the QCD vacuum. The $U_A(1)$ symmetry, however, has been an issue

for quite some time. If the $U_A(1)$ were intact, the light hadrons would occur in degenerate parity doublets, which is not the case. Furthermore, if $U_A(1)$ were spontaneously broken, there should be a (nearly) massless (or at least light) pseudoscalar flavor singlet meson, the "would-be" Goldstone boson of spontaneous $U_A(1)$ symmetry breaking. The only candidate is the $\eta'$, which, however, is by far too heavy to qualify for this particle.

It was first shown by Adler [1], Bell and Jackiw [2] that the $U_A(1)$ is anomalously broken, i.e. broken by quantum effects. The anomalous breaking of the global $U_A(1)$ symmetry manifests itself in the non-invariance of the fermionic integration measure [3,4] and results in the well-known axial anomaly, which for massless quarks reads

$$\partial_\mu j_5^\mu(x) = 2\,N_f\,q(x). \tag{5.2}$$

Here, $j_5^\mu(x) = \bar\psi(x)\gamma^\mu\gamma_5\psi(x)$ is the axial current and

$$q(x) = \frac{g^2}{32\pi^2}\,F^a_{\mu\nu}(x)\widetilde{F}^{\mu\nu}_a(x) \tag{5.3}$$

is the topological charge density in Minkowski space, with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,f^{abc}A^b_\mu A^c_\nu \tag{5.4a}$$

$$\widetilde{F}^{\mu\nu}_a = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}F^a_{\rho\sigma} \tag{5.4b}$$

being the field strength tensor and its dual ($f^{abc}$ are the structure constants of the $\mathfrak{su}(N_\mathrm{c})$ algebra and $g$ is the coupling constant; we use colour sub- and superscripts indiscriminately). Adler and Bardeen showed [5] that Eq. (5.2) does not recieve corrections from higher-order diagrams. Although $q(x)$ is a total divergence

$$F^a_{\mu\nu}\,\widetilde{F}^{\mu\nu}_a = 2\partial_\mu K^\mu, \tag{5.5}$$

where

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}\left[A^a_\nu\partial_\rho A^a_\sigma + \frac{1}{3}\,g f^{abc}\,A^a_\nu A^b_\rho A^c_\sigma\right] \tag{5.6}$$

is the topological current, there are topologically non-trivial field configurations (in Euclidean space) such as instantons [6,7], magnetic monopoles [8] or center vortices [9,10], for which the topological charge

$$Q = \int \mathrm{d}^4x_\mathrm{E}\;q(x) \tag{5.7}$$

is non-zero (for smooth field configurations of finite Euclidean action, $Q$ is integer-valued). As a consequence of the existence of these field configurations, the axial charge

$$Q_5 = \int \mathrm{d}^3x\;j_5^0(x) \tag{5.8}$$

is not conserved [11,12].

Using large-$N_\mathrm{c}$ arguments, Witten [13] and Veneziano [14] showed that the axial anomaly provides a mass term for the pseudoscalar flavor singlet meson given by

$$m^2_{\eta'} + m^2_\eta - 2m^2_K = \frac{2N_f}{F^2_\pi}\,\chi, \tag{5.9}$$

where $F_\pi \simeq 93$ MeV is the pion decay constant and

$$\chi = \int \mathrm{d}^4 x_\mathrm{E} \, \langle 0| \, q(x) \, q(0) \, |0\rangle \tag{5.10}$$

is the topological susceptibility, which is a purely gluonic quantity (in Minkowski space Eq. (5.10) has an additional factor $-\mathrm{i}$). Since this quantity is defined as vacuum expectation value it is clear that its evaluation requires non-perturbative methods. Indeed, in perturbation theory the topological susceptibility vanishes to all orders. This quantity has been calculated on the lattice (see Refs. [15,16] for recent calculations) and the results are compatible with the prediction of the Witten–Veneziano formula Eq. (5.9)

$$\chi \simeq (180 \text{ MeV})^4 \tag{5.11}$$

using the experimental data for the meson masses and $F_\pi$ as input.

Like the string tension [17], the topological susceptibility seems to be dominated by center vortices [18]. In fact, a center vortex model of the infrared sector of Yang–Mills theory [19] yields a value for $\chi$ compatible with lattice results [20].

Topologically non-trivial Euclidean field configurations such as instantons and center vortices describe quantum tunneling between topologically different Yang–Mills vacua [21], see Eq. (5.12) below. Like in quantum mechanics, this quantum tunneling is fully accounted for by the solution of the Yang–Mills Schrödinger equation. Recently, progress has been made in determining the vacuum wave functional by a variational solution of the Yang–Mills Schrödinger equation in Coulomb gauge [22–30]. There is good evidence to believe that the obtained wave functional contains the essential infrared physics: the absence of gluons from the physical spectrum in the infrared [25], a linearly rising potential for static colour charges [27] and a perimeter law for the 't Hooft loop [28]. In the present paper we use the Yang–Mills wave functional determined in Refs. [25,27] to calculate the topological susceptibility given by Eq. (5.10).

The organization of the paper is as follows: In the next section we briefly review the $\theta$-vacuum in the Hamiltonian approach. In section 3 we derive the expressions for the topological susceptibility in the Hamiltonian approach in Coulomb gauge and evaluate the relevant matrix elements. Our numerical results are presented in section 4. Some concluding remarks are given in section 5.

## 5.2 The $\theta$-vacuum in the canonical quantization approach

Consider Yang–Mills theory in the Weyl gauge $A_0^a = 0$. The classical (time-independent) vacuum configurations are pure gauge spatial fields

$$A_i = A_i^a \, t_a = \frac{\mathrm{i}}{g} \, U \partial_i U^\dagger, \tag{5.12}$$

where $t_a$ are the Hermitian generators of the gauge group. Imposing the usual boundary condition that the gauge function $U(\mathbf{x}) \in SU(N_\mathrm{c})$ approaches a unique value for $|\mathbf{x}| \to \infty$ (independent of the direction $\hat{\mathbf{x}}$) compactifies $\mathbb{R}^3$ to $S_3$ and consequently the $U(\mathbf{x})$ can be classified according to the winding number $n[U] \in \Pi_3(S_3)$ [21]. The classical vacuum configurations Eq. (5.12) belonging to different winding numbers are separated by infinite potential barriers. In the quantum theory, tunnelling between the different classical vacua

occurs and in a semiclassical picture, the barrier penetration is described by instantons, (space-)time-dependent solutions of the classical Euclidean Yang–Mills equation of motion, which interpolate between vacuum configurations differing in the winding number by $\pm 1$. We will not resort here to a semiclassical description but instead approximately solve the Schrödinger equation, which fully accounts for the quantum tunnelling.

In Weyl gauge the physical coordinates are the spatial gauge fields $A_i^a(x)$ and the corresponding canonical conjugate momenta are given by the chromoelectric field $E_i^a(x) = F_{0i}^a(x)$. In canonical quantization the electric field is promoted to the momentum operator $\Pi_i^a(x) = -\mathrm{i}\delta/\delta A_i^a(x)$ satisfying canonical commutation relations, and the Yang–Mills Hamiltonian reads

$$H = \frac{1}{2}\int \mathrm{d}^3x \,\left[\left(\Pi_i^a(\mathbf{x})\right)^2 + \left(B_i^a(\mathbf{x})\right)^2\right], \tag{5.13}$$

where

$$B_i^a = \frac{1}{2}\,\varepsilon_{ijk}\,F_{jk}^a = \varepsilon_{ijk}\left[\partial_j A_k^a + \frac{g}{2} f^{abc} A_j^b A_k^c\right] \tag{5.14}$$

is the non-abelian magnetic field.

In Weyl gauge, Gauss's law is lost from the equations of motion and has to be imposed as a constraint on the wave functional

$$\hat{D}_i^{ab}\,\Pi_i^b(\mathbf{x})\,\Psi[A] = -g\,\rho_{\mathrm{ext}}^a(\mathbf{x})\,\Psi[A]. \tag{5.15}$$

Here,

$$\hat{D}_i^{ab} = \delta^{ab}\partial_i + g\hat{A}_i^{ab}, \qquad \hat{A}^{ab} = f^{acb}A^c \tag{5.16}$$

is the covariant derivative in the adjoint representation of the gauge group and $\rho_{\mathrm{ext}}^a(\mathbf{x})$ denotes the external colour charge density of the matter fields. The operator $\hat{D}\Pi$ on the left-hand side of Eq. (5.15) is the generator of time-independent small gauge transformations ($n[U] = 0$). In the absence of external charges, $\rho_{\mathrm{ext}}^a(\mathbf{x}) = 0$, Gauss's law requires the wave functional to be invariant under small gauge transformations only, while under large gauge transformation it needs to be invariant only up to a phase [31,32]

$$\Psi_\theta[A^U] = \mathrm{e}^{-\mathrm{i}\theta n[U]}\Psi_\theta[A]. \tag{5.17}$$

Here, $\theta$ is a free real parameter, which characterizes the vacuum wave functional and is called the vacuum angle. Since $n[U] \in \mathbb{Z}$, the wave functional of the $\theta$-vacuum Eq. (5.17) has the property

$$\Psi_{\theta+2\pi}[A] = \Psi_\theta[A], \tag{5.18}$$

which qualifies $\theta$ as an angle variable. $\theta$ is not known a priori and is not determined by the theory itself but must be fixed by experiment. One measurable effect of $\theta$ would be a non-vanishing neutron electric dipole moment [33]. Current measurements [34] restrict $\theta$ to the extremely small value $|\theta| \leq 10^{-10}$.

The transformation property Eq. (5.17) can be realized by the ansatz

$$\Psi_\theta[A] = \mathrm{e}^{-\mathrm{i}\theta W[A]}\phi[A], \tag{5.19}$$

where $\phi[A]$ is a gauge invariant wave functional, $\phi[A^U] = \phi[A]$, and

$$W[A] = \frac{g^2}{16\pi^2}\int \mathrm{d}^3x\, K^0(x) = \frac{g^2}{16\pi^2}\,\varepsilon_{ijk}\int \mathrm{d}^3x\,\left[A_i^a\partial_j A_k^a + \frac{g}{3}\,f^{abc}A_i^a A_j^b A_k^c\right] \tag{5.20}$$

is the Chern–Simons action, which changes under gauge transformations by the winding number $n[U]$

$$W[A^U] = W[A] + n[U]. \tag{5.21}$$

The wave functional Eq. (5.19) does, however, not fulfill Eq. (5.18). In the appendix we show how Eq. (5.19) has to be modified to promote $\theta$ to an angle variable. We also show there that Eq. (5.19) is a correct wave functional when $\theta$ is restricted to $[0, 2\pi)$. In the following, the cyclic property Eq. (5.18) of the vacuum wave functional will be irrelevant, since we are anyway interested only in infinitesimally small $\theta$, see Eq. (5.22) below. Therefore, we can use the simpler wave functional Eq. (5.19).

The quantity of interest is the topological susceptibility Eq. (5.10) which in the Hamiltonian approach can be defined by [13]

$$V\chi = \left.\frac{\mathrm{d}^2\langle H\rangle_\theta}{\mathrm{d}\theta^2}\right|_{\theta=0}^{\text{no quarks}}, \tag{5.22}$$

where $V$ is the spatial volume and

$$\langle H\rangle_\theta = \langle \Psi_\theta |\, H \,| \Psi_\theta\rangle, \qquad \langle \Psi_\theta | \Psi_\theta\rangle = 1 \tag{5.23}$$

is the expectation value of the Yang–Mills Hamiltonian in the $\theta$-vacuum. To evaluate the $\theta$-dependence of $\langle H\rangle_\theta$, the following identity will be useful

$$\frac{\delta W[A]}{\delta A_i^a(x)} = \frac{g^2}{8\pi^2}\, B_i^a(x), \tag{5.24}$$

where $B_i^a$ is the magnetic field Eq. (5.14). Obviously, the $\theta$-phase in Eq. (5.19) can only contribute to the kinetic term of the Yang–Mills Hamiltonian Eq. (5.13). With Eq. (5.24) we find from Eq. (5.19)

$$\Pi_i^a \Psi_\theta[A] = \mathrm{e}^{-\mathrm{i}\theta W[A]} \left(\Pi_i^a - \theta\, \frac{g^2}{8\pi^2}\, B_i^a\right) \phi[A]. \tag{5.25}$$

Inserting this relation into Eq. (5.15) and using the Bianchi identity

$$\hat{D}_i^{ab} B_i^b(x) = 0 \tag{5.26}$$

we find that the wave functional $\phi[A]$ satisfies the same Gauss law as $\Psi_\theta[A]$

$$\hat{D}_i^{ab}\, \Pi_i^b\, \phi[A] = -g\, \rho_{\text{ext}}^a\, \phi[A]. \tag{5.27}$$

To make contact with previous results obtained in the Hamiltonian approach to Yang–Mills theory (see Refs. [25,27]), it is convenient to work with the $\theta$-independent wave functional $\phi[A]$ and absorb the $\theta$-dependence into the Hamiltonian by defining

$$\langle \Psi_\theta |\, H \,| \Psi_\theta\rangle = \langle \phi |\, H_\theta \,| \phi\rangle. \tag{5.28}$$

Using Eq. (5.25) we find

$$H_\theta = \frac{1}{2}\int \left(\Pi_i^a - \theta\, \frac{g^2}{8\pi^2}\, B_i^a\right)^2 + \frac{1}{2}\int \left(B_i^a\right)^2. \tag{5.29}$$

An alternative way to arrive at this Hamiltonian is to add the topological $\theta$-term directly to the original classical Lagrangian

$$\mathcal{L} = -\frac{1}{4}\, F^a_{\mu\nu}\, F^{\mu\nu}_a + \theta\, \frac{g^2}{32\pi^2}\, F^a_{\mu\nu} \widetilde{F}^{\mu\nu}_a = \frac{1}{2}\big(\mathbf{E}^2 - \mathbf{B}^2\big) + \theta\, \frac{g^2}{8\pi^2}\, \mathbf{E}\cdot\mathbf{B}. \tag{5.30}$$

The $\theta$-term is a total derivative and does not contribute to the classical equation of motion. It does, however, change the canonical momentum from $\Pi^a_i = F^a_{0i}$ to

$$\Pi^a_i = F^a_{0i} + \theta\, \frac{g^2}{8\pi^2}\, B^a_i \tag{5.31}$$

and after canonical quantization in Weyl gauge one finds again the Hamiltonian $H_\theta$ [Eq. (5.29)].

Instead of working (for $\rho^a_{\rm ext}(\mathbf{x}) = 0$) with gauge invariant wave functionals $\phi[A]$, it is more convenient to explicitly resolve Gauss's law Eq. (5.27) by fixing the gauge. For this purpose Coulomb gauge $\partial_i A^a_i = 0$ is particularly convenient and will be used in the following. In Coulomb gauge, the gauge field is transverse $A = A^\perp$ but this is not true for the momentum operator. We split the momentum operator into longitudinal and transverse parts $\Pi = \Pi^\perp + \Pi^\parallel$, where $\Pi^\perp = -\mathrm{i}\delta/\delta A^\perp$. The latter satisfies the canonical commutation relation for transverse fields

$$\big[A^{\perp a}_i(\mathbf{x}), \Pi^{\perp b}_j(\mathbf{y})\big] = \mathrm{i}\,\delta^{ab}\, t_{ij}(\mathbf{x})\, \delta(\mathbf{x}-\mathbf{y}), \tag{5.32}$$

where $t_{ij}(\mathbf{x}) = \delta_{ij} - \partial_i\partial_j/\partial^2$ is the transverse projector.

Gauss's law Eq. (5.27) can be solved for the longitudinal part $\Pi^\parallel$ in the standard fashion yielding

$$\Pi^\parallel\,|\phi\rangle = -g\partial(-\hat{D}\partial)^{-1}(\rho_{\rm ext} + \rho_g)\,|\phi\rangle, \tag{5.33}$$

where $\rho^a_g = \hat{A}^{\perp ab}_i \Pi^{\perp b}_i$ is the colour charge density of the gluons and $(-\hat{D}\partial)$ is the Faddeev–Popov kernel in Coulomb gauge. With the aid of Eq. (5.33), one derives from Eq. (5.29) the gauge fixed Hamiltonian of the $\theta$-vacuum (by considering $\langle\phi|\, H_\theta\,|\phi\rangle$ and using integration by parts in the kinetic term). This yields[1]

$$H_\theta = H_0 + \theta\, \frac{g^2}{8\pi^2}\, H_1 + \left(\theta\, \frac{g^2}{8\pi^2}\right)^2 H_2\,. \tag{5.34}$$

where

$$\begin{aligned} H_0 = \frac{1}{2}\int \mathrm{d}^3x\, \big[\mathcal{J}_A^{-1}\Pi^a_i(\mathbf{x})\mathcal{J}_A\Pi^a_i(\mathbf{x}) + B^a_i(\mathbf{x})\, B^a_i(\mathbf{x})\big] \\ + \frac{g^2}{2}\int \mathrm{d}^3x\, \mathrm{d}^3y\, \mathcal{J}_A^{-1}\, \rho^a(\mathbf{x})\, \mathcal{J}_A\, F^{ab}_A(\mathbf{x},\mathbf{y})\rho^b(\mathbf{y}) \end{aligned} \tag{5.35}$$

is the usual Coulomb gauge fixed Hamiltonian [35] for $\theta = 0$ and the $\theta$-dependent terms are given by

$$\begin{aligned} H_1 = -\frac{1}{2}\int \mathrm{d}^3x\, \big[B^a_i(\mathbf{x})\Pi^a_i(\mathbf{x}) + \mathcal{J}_A^{-1}\Pi^a_i(\mathbf{x})\mathcal{J}_A B^a_i(\mathbf{x})\big] \\ + \frac{1}{2}\int \mathrm{d}^3x\, \mathrm{d}^3y\, \Big\{G^{ab}_A(\mathbf{x},\mathbf{y})\partial^x_i B^a_i(\mathbf{x}) g\rho^b(\mathbf{y}) + \mathcal{J}_A^{-1} g\rho^a(\mathbf{x})\mathcal{J}_A G^{ab}_A(\mathbf{x},\mathbf{y})\partial^y_i B^b_i(\mathbf{y})\Big\} \end{aligned} \tag{5.36}$$

[1] In the following, all field and momentum operators are transverse and we will omit the symbol $\perp$.

and

$$H_2 = \frac{1}{2}\int \mathrm{d}^3x\; B_i^a(\mathbf{x})B_i^a(\mathbf{x}). \tag{5.37}$$

In the above expressions, $\mathcal{J}_A = \mathrm{Det}(-\partial_i\hat{D}_i)$ is the Faddeev–Popov determinant, $G_A$ is the Green function of the Faddeev–Popov operator

$$-\partial_i\hat{D}_i^{ab}(\mathbf{x})\, G_A^{bc}(\mathbf{x},\mathbf{y}) = \delta^{ac}\;\delta(\mathbf{x}-\mathbf{y}), \tag{5.38}$$

$F_A$ denotes the Coulomb operator

$$F_A^{ab}(\mathbf{x},\mathbf{y}) = \Big[(-\partial_i\hat{D}_i)^{-1}(-\partial^2)(-\partial_j\hat{D}_j)^{-1}\Big]^{ab}_{\mathbf{x},\mathbf{y}} \tag{5.39}$$

and $\rho$ is the sum of external and dynamical colour charge densities

$$\rho^a = \rho^a_{\mathrm{ext}} + \rho^a_g = \rho^a_{\mathrm{ext}} + f^{abc}A_i^b\,\Pi_i^c. \tag{5.40}$$

For the present purpose, the evaluation of the topological susceptibility $\chi$ (which is entirely defined in the gluon sector), the external charges are not needed and we will put $\rho^a_{\mathrm{ext}} = 0$ in the following. The Faddeev–Popov determinant $\mathcal{J}_A$ represents the Jacobian of the transformation from the (flat) non gauge-fixed configuration space to the (curved) space of the Coulomb gauge fixed fields. In particular, this Jacobian enters the integration measure of the scalar product of wave functions

$$(\Phi_1,\Phi_2) = \int \mathcal{D}A\;\mathcal{J}_A\;\Phi_1^*[A]\;\Phi_2[A]. \tag{5.41}$$

The integration of transverse field configurations extends over the first Gribov region $\Omega$ [36], allowing the surface terms to be discarded in integration by parts. Although the integration should be restricted to the fundamental modular region $\Lambda \subset \Omega$ [37], there is evidence that integration over $\Omega$ yields the same expectation values [38].

Following Ref. [25] we introduce the "radial" wave functional

$$\widetilde{\Phi}[A] = \mathcal{J}_A^{1/2}\;\Phi[A] \tag{5.42}$$

which removes the Faddeev–Popov determinant from the integration measure of the scalar product. Matrix elements of observables $O[A,\Pi]$ can then be expressed as

$$\int \mathcal{D}A\;\mathcal{J}_A\;\Phi_1^*\,O\,\Phi_2 = \int \mathcal{D}A\;\widetilde{\Phi}_1^*\,\widetilde{O}\,\widetilde{\Phi}_2\,, \tag{5.43}$$

where we have introduced the transformed operator

$$\widetilde{O}[A,\Pi] = \mathcal{J}_A^{1/2}\;O[A,\Pi]\mathcal{J}_A^{-1/2} = O[A,\widetilde{\Pi}]. \tag{5.44}$$

An operator $O[A]$ depending only on the field variable $A_i^a(\mathbf{x})$ is obviously not changed by this transformation, while the momentum operator transforms as

$$\widetilde{\Pi}_i^a(\mathbf{x}) = \Pi_i^a(\mathbf{x}) + \frac{\mathrm{i}}{2}\,\frac{\delta\ln\mathcal{J}_A}{\delta A_i^a(\mathbf{x})}\,. \tag{5.45}$$

The transformed Hamilton operator $\widetilde{H}_\theta = \mathcal{J}_A^{1/2}H_\theta\mathcal{J}_A^{-1/2}$ is obtained from $H_\theta$ by replacing $\Pi$ by $\widetilde{\Pi}$. $\widetilde{H}_0$ was explicitly given in Ref. [25]. Furthermore, since $B_i^a(\mathbf{x})$ is a function of the field variable only, $H_2$ does not change ($\widetilde{H}_2 = H_2$). The explicit expression for $\widetilde{H}_1$ is given in the next section.

## 5.3 Matrix elements for the topological susceptibility

We are interested here in the topological susceptibility, Eq. (5.22). Since this quantity is defined as second derivative of $\langle H_\theta \rangle$ at $\theta = 0$, it is sufficient to calculate $\langle H_\theta \rangle$ up to second order in $\theta$. This requires to treat $H_2$ in first-order and $\widetilde{H}_1$ up to second-order perturbation theory in $\theta$. We then find

$$V\,\chi = 2\left(\frac{g^2}{8\pi^2}\right)^2 \left[\langle 0|\, H_2\, |0\rangle - \sum_{n\neq 0} \frac{|\langle 0|\, \widetilde{H}_1\, |n\rangle|^2}{E_n}\right] \tag{5.46}$$

where $\{|n\rangle\}$ denotes the set of eigenstates of $\widetilde{H}_0$. We use here a Dirac notation for the radial wave functionals Eq. (5.42), $\langle A|0\rangle = \widetilde{\Phi}_0[A]$.

For the unperturbed Hamiltonian $\widetilde{H}_0$, we use the variational results obtained in Refs. [25,27]. For the (radial) vacuum wave functional the following Gaussian form was chosen

$$\widetilde{\Phi}_0[A] = \langle A|0\rangle = \mathcal{N} \exp\left[-\frac{1}{2}\int \mathrm{d}^3x\ \mathrm{d}^3y\ A_i^a(\mathbf{x})\,\omega_{ij}(\mathbf{x},\mathbf{y})\,A_j^a(\mathbf{y})\right], \tag{5.47}$$

where $\omega_{ij}(\mathbf{x},\mathbf{y}) = t_{ij}(\mathbf{x})\omega(\mathbf{x},\mathbf{y})$ and $\omega(\mathbf{x},\mathbf{y})$ is an integration kernel determined by minimization of the energy density. By means of Wick's theorem we can express expectation values of powers of field operators by the gluon propagator

$$\langle 0|\, A_i^a(\mathbf{x})\, A_j^b(\mathbf{y})\, |0\rangle = \frac{1}{2}\,\delta^{ab}\,\omega_{ij}^{-1}(\mathbf{x},\mathbf{y}). \tag{5.48}$$

Note that $\omega(\mathbf{x},\mathbf{y})$ depends only on $|\mathbf{x}-\mathbf{y}|$ and by Eq. (5.48) its Fourier transform represents the single quasi-gluon energy.

The vacuum wave functional $|0\rangle$ [Eq. (5.47)] is annihilated $a_i^a(x)\,|0\rangle = 0$ by the operator

$$a_i^a = \frac{1}{\sqrt{2}}\left[\omega_{ij}^{1/2} A_j^a + \mathrm{i}\,\omega_{ij}^{-1/2}\Pi_j^a\right], \tag{5.49}$$

which is the annihilation operator of a quasi-gluon with energy $\omega$. Here and in the following we suppress the explicit spatial dependence of the involved quantities and include the spatial coordinates in the Lorentz indices, so that contracted Lorentz indices imply integration over the spatial coordinate. The corresponding creation operator reads

$$a_i^{a\dagger} = \frac{1}{\sqrt{2}}\left[\omega_{ij}^{1/2} A_j^a - \mathrm{i}\,\omega_{ij}^{-1/2}\Pi_j^a\right], \tag{5.50}$$

and from Eq. (5.32) follows that these operators satisfy the usual Bose commutation relation

$$\left[a_i^a(\mathbf{x}), a_j^{b\dagger}(\mathbf{y})\right] = \delta^{ab}\, t_{ij}(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{y}) \tag{5.51}$$

(temporarily restoring the explicit spatial dependence for clarity).

By repeated application of the creation operator $a^\dagger$ [Eq. (5.50)] on the vacuum Eq. (5.47) a complete basis for the gluon Fock-space of quasi-gluons is generated

$$|n\rangle = \mathcal{N}_n \prod_{k=1}^{n} a_{i_k}^{a_k\dagger}(\mathbf{x}_k)\,|0\rangle. \tag{5.52}$$

For a proper normalization of these quasi-particle states the additional normalization factor $\mathcal{N}_n$ is required when two or more indices take the same value, which can occur due to the bosonic character of these quasi-particle excitations.

With a complete basis of the Yang–Mills Hilbert space at our dipsosal we can explicitly carry out the perturbative calculations in Eq. (5.46). The expectation value of $H_2$ can be straightforwardly evaluated, yielding

$$\begin{aligned}\langle 0|\, H_2\, |0\rangle = \frac{N_c^2-1}{2}\, V \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \frac{\mathbf{k}^2}{\omega(\mathbf{k})} \\ + g^2\, \frac{N_c(N_c^2-1)}{16}\, V \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\mathrm{d}^3q}{(2\pi)^3}\, \frac{3-(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})^2}{\omega(\mathbf{k})\,\omega(\mathbf{q})}\,.\end{aligned} \tag{5.53}$$

The second term in Eq. (5.46) contains a sum over an infinite number of states with an arbitrary large number of quasi-gluons and we have to resort to some approximations. Motivated by the variational calculation for $\theta = 0$ [25,27] we restrict ourselves to terms involving up to two loops in the energy. Under this assumption, $\widetilde{H}_1$ [Eq. (5.36)] is given by

$$\widetilde{H}_1 = -\int \mathrm{d}^3x\, \Pi_i^a(\mathbf{x})\, B_i^a(\mathbf{x}) + \int \mathrm{d}^3x\ \mathrm{d}^3y\, G_A^{ab}(\mathbf{x},\mathbf{y})\, \partial_i^x B_i^a(\mathbf{x})\, g\hat{A}_j^{bc}(\mathbf{y})\, \Pi_j^c(\mathbf{y}), \tag{5.54}$$

and we can use the following factorization in the matrix elements of the second term of Eq. (5.54)

$$\langle 0|\, G_A\, (\partial B)\, (g\hat{A}\,\Pi)\, |n\rangle \simeq \langle 0|\, G_A\, |0\rangle \langle 0|\, (\partial B)\, (g\hat{A}\,\Pi)\, |n\rangle, \tag{5.55}$$

where $\langle G_A\rangle$ is the ghost propagator. With these approximations, the relevant contributions to the second term in Eq. (5.46)

$$-\sum_{n\neq 0} \frac{|\langle 0|\, \widetilde{H}_1\, |n\rangle|^2}{E_n} =: \mathcal{M}_2 + \mathcal{M}_3 \tag{5.56}$$

come from two quasi-gluon states, $n = 2$

$$\mathcal{M}_2 = -\frac{1}{2!} \sum_{1,2} \frac{|\langle 0|\, B\Pi\, |2\rangle - \langle G\rangle \langle 0|\, \partial B g\hat{A}\Pi\, |2\rangle|^2}{E_2}\,, \tag{5.57}$$

and from the 3-quasi-gluon states, $n = 3$

$$\mathcal{M}_3 = -\frac{1}{3!} \sum_{1,2,3} \frac{|\langle 0|\, B\Pi\, |3\rangle|^2}{E_3}\,. \tag{5.58}$$

The prefactors 1/2! and 1/3! take the normalization of the states Eq. (5.52) into account and avoid multiple counting. $E_2$ and $E_3$, respectively, denote the energies of the two- and three-quasi-gluon states and accordingly are given by sums of, respectively, two and three $\omega$'s.

The matrix elements in Eqs. (5.57), (5.58) can be evaluated by means of Wick's theorem, yielding

$$\mathcal{M}_2 = -\frac{N_c^2-1}{2}\,V\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{\mathbf{k}^2}{\omega(\mathbf{k})}+ \\ -g^2\,\frac{N_c(N_c^2-1)}{2}\,V\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{\mathbf{k}^2}{\omega(\mathbf{k})}\,G(\mathbf{k}+\mathbf{q}) \\ \times\left(\frac{1}{\omega(\mathbf{k})}+\frac{1}{\omega(\mathbf{q})}\right)\left[\frac{1+(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})^2}{2}+\frac{\mathbf{k}\cdot\mathbf{q}}{\mathbf{k}^2}\right], \quad (5.59)$$

$$\mathcal{M}_3 = -g^2\,\frac{N_c(N_c^2-1)}{48}\,V\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{\omega(\mathbf{k})+\omega(\mathbf{q})+\omega(\mathbf{p})}{\omega(\mathbf{k})\,\omega(\mathbf{q})\,\omega(\mathbf{p})} \\ \times\Big[2-2(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})\Big]\,(2\pi)^3\delta(\mathbf{k}+\mathbf{q}+\mathbf{p}). \quad (5.60)$$

Within our approximations the expression in the bracket in Eq. (5.46) is given by the sum of Eqs. (5.53), (5.59) and (5.60). The term involving $\langle G\rangle^2$ in Eq. (5.57) does not appear in Eq. (5.59) since it involves three loops. It is worth noting that the first term in Eq. (5.59) cancels the first term in Eq. (5.53). This cancellation is due to the fact that the Gaussian wave functional is exact in an Abelian theory and the two terms are actually the leading order contribution to the topological susceptibility in perturbation theory in powers of $g$, where $\chi$ vanishes identically. In Ref. [39] it has been explicitly shown that this cancellation occurs also in next-to-leading order.

The integrals in Eqs. (5.53), (5.59) and (5.60) are UV divergent. Since the topological susceptibility $\chi$ vanishes to any (finite) order perturbation theory in $g$, in principle, all UV-divergences should cancel. However, the approach [24,25] we are using is non-perturbative and, due to the approximations involved, does not include all terms of a given power in $g$. As a consequence, there is a mismatch of UV-singularities and our expression for $\chi$ is UV-divergent. To remove these spurious UV-singularities we subtract from all propagators the corresponding tree-level form, i.e. we make the following replacements

$$\omega^{-1}(\mathbf{k}) \to \omega_s^{-1}(\mathbf{k}) = \omega^{-1}(\mathbf{k}) - 1/\sqrt{\mathbf{k}^2}\,, \quad (5.61a)$$

$$G(\mathbf{k}) \to G_s(\mathbf{k}) = G(\mathbf{k}) - 1/\mathbf{k}^2. \quad (5.61b)$$

This is in the spirit of the zero-momentum subtraction scheme. The replacement Eq. (5.61) makes the integrals Eqs. (5.53), (5.59) and (5.60) convergent. Using the symmetry of the integrands we can finally cast the expression for the topological susceptibility into the form

$$\chi = g^2\left(\frac{g^2}{8\pi^2}\right)^2 N_c(N_c^2-1)(I_1+I_2), \quad (5.62)$$

where

$$I_1 = \frac{1}{4}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{\mathrm{d}^3q}{(2\pi)^3}\,\omega_s^{-1}(\mathbf{k})\,\omega_s^{-1}(\mathbf{q})\,\left[1-\mathbf{k}^2\,\frac{1-(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})^2}{(\mathbf{k}+\mathbf{q})^2}\right] \quad (5.63a)$$

$$I_2 = -\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{\mathrm{d}^3q}{(2\pi)^3}\,k^2\,\omega_s^{-1}(\mathbf{k})\,\left[\omega_s^{-1}(\mathbf{k})+\omega_s^{-1}(\mathbf{q})\right]G_s(\mathbf{k}+\mathbf{q})\left[\frac{1+(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})^2}{2}+\frac{\mathbf{q}\cdot\mathbf{k}}{\mathbf{k}^2}\right]. \quad (5.63b)$$

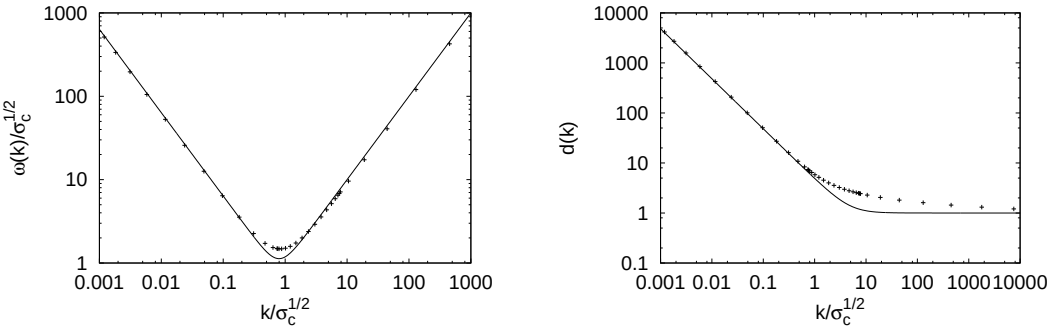


Figure 5.1: Comparison between the numerical results of Ref. [27] (crosses) and the parametrizations Eq. (5.64) (lines) for the inverse gluon propagator (top panel) and the ghost form factor (bottom panel).

Eq. (5.63a) follows from the sum of Eq. (5.60) and the second term in Eq. (5.53) while Eq. (5.63b) follows from the second term in Eq. (5.59). The angular integration in Eq. (5.63a) can be performed analytically.

## 5.4 Results

The expressions for the topological susceptibility, Eqs. (5.62) and (5.63), depend via the gluon and ghost propagators, $\omega^{-1}$ and $G$, on the vacuum properties, i.e. on the vacuum wave functional. We will use here the ghost and gluon propagators determined in Ref. [27] as input. Furthermore, to simplify our calculations the numerical solutions obtained in Ref. [27] for the gluon and ghost propagators were fitted by the following ansätze

$$\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + \frac{m^4}{\mathbf{k}^2}}\,, \tag{5.64a}$$

$$G(\mathbf{k}) = \frac{1}{\mathbf{k}^2}\sqrt{1 + \frac{M^2}{g^2\,\mathbf{k}^2}}\,, \tag{5.64b}$$

($g$ is the coupling constant). The parametrization Eq. (5.64a) of the gluon energy was already assumed heuristically by Gribov [36]. A factor $g$ was included in the parametrization of the ghost propagator Eq. (5.64b) in order to facilitate the fitting to the numerical results of Ref. [27], where a factor $g$ was included in the definition of the ghost form factor $d(\mathbf{k})$ [2]

$$G(\mathbf{k}) = \frac{d(\mathbf{k})}{g\,\mathbf{k}^2}\,. \tag{5.65}$$

The numerical solutions of Ref. [27] and their parametrizations by Eq. (5.64) are shown in Fig. 5.1. The integrals Eq. (5.63) entering the topological susceptibility Eq. (5.62) recieve their dominant contributions from the infrared momentum regime, where the parametrizations Eq. (5.64) give a perfect fit to the numerical solutions. The infrared part of the

[2] The inverse ghost form factor $gd^{-1}(\mathbf{k})$ has the meaning of the dielectric function of the Yang–Mills vacuum [40].

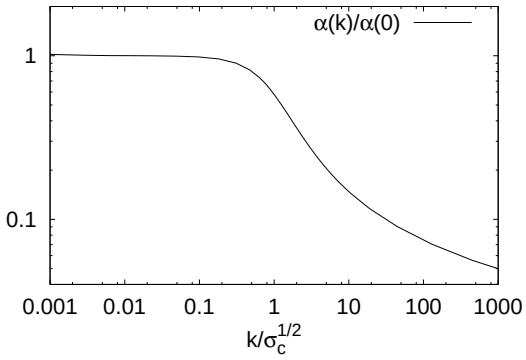


Figure 5.2: Running coupling as calculated in [27].

parametrizations Eq. (5.64)

$$\omega_{\rm IR}(\mathbf{k}) = \frac{m^2}{\sqrt{\mathbf{k}^2}}\,, \quad d_{\rm IR}(\mathbf{k}) = \frac{M}{\sqrt{\mathbf{k}^2}} \tag{5.66}$$

was fitted to the infrared behaviour of the gluon and ghost propagators found in Ref. [27] by solving the corresponding Dyson–Schwinger equations (DSEs). In the Hamiltonian approach in Coulomb gauge the physical scale is the Coulomb string tension $\sigma_{\rm C}$, i.e. the coefficient of the linear term in the non-abelian Coulomb potential. In units of the Coulomb string tension $\sigma_{\rm C}$ the fit of Eq. (5.66) to the numerical solutions of the DSEs yields $m^2 = 0.614$ and $M = 4.97$, while from the (analytic) infrared analysis of the DSEs [25,27,41] one extracts the values $m^2 = 2/\pi \simeq 0.637$ and $M = \sqrt{8\pi} \simeq 5.01$.

With the definition Eq. (5.65) of the ghost form factor in the variational calculation of Refs. [22–27] the coupling constant $g$ drops out from the DSEs and the value of the coupling constant never had to be specified. Contrary to this, the topological susceptibility Eq. (5.62) explicitly contains the coupling constant. In principle, after a complete renormalization procedure, in all physical (renormalized) quantities the coupling constant should be replaced by the running one defined in a renormalization group invariant way. Such a complete renormalization program is not yet feasible, although some progress in this direction has been made.[3] Fortunately the running couling constant calculated in Ref. [27] in the Hamiltonian approach in Coulomb gauge (see Fig. 5.2) has a very weak momentum dependence in the infrared regime. It basically stays constant below the infrared scale $\sqrt{\sigma_{\rm C}}$. It is this low-momentum regime which gives the major contribution to the integrals Eq. (5.63). We will therefore use the plateau value of the running coupling constant. Using the definition of the non-perturbative running coupling given in Ref. [43], its value at zero momentum is [41]

$$\alpha_s(0) = \frac{g^2(0)}{4\pi} = \frac{16\pi}{3N_{\rm c}}\,. \tag{5.67}$$

Numerical (Gauss–Legendre) evaluation of the integrals Eq. (5.63) yields

$$I_1 = (0.077\sqrt{\sigma_{\rm C}})^4, \quad I_2 = (0.021\sqrt{\sigma_{\rm C}})^4. \tag{5.68}$$

[3] The counterterms required for the renormalization have been identified [30,42].

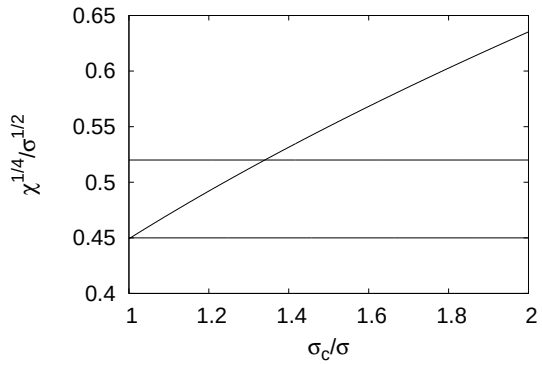


Figure 5.3: Value of $\chi^{\frac{1}{4}}/\sqrt{\sigma}$ depending on the ratio $\sigma_{\rm C}/\sigma$. The two horizontal lines limit the range of the lattice results.

Then we find for the topological susceptibility for $N_{\rm c} = 2$

$$\frac{\chi^{\frac{1}{4}}}{\sqrt{\sigma_{\rm C}}} = 0.45\,. \tag{5.69}$$

It was shown in Ref. [44] that $\sigma_{\rm C}$ is an upper bound for the string tension $\sigma$ extracted from the Wilson loop. Lattice calculations performed in Ref. [45] for $SU(2)$ show that $\sigma_{\rm C} \simeq 1.5\,\sigma$, while Ref. [46] seems to suggest an even larger value for $\sigma_{\rm C}$. In Fig. 5.3 we present our numerical result for the topological susceptibility as a function of $\sigma_{\rm C}/\sigma$. For $1 < \sigma_{\rm C}/\sigma < 1.33$, $\chi$ is inside the range predicted by the lattice data [47–50]. Choosing $\sigma_{\rm C} = 1.5\,\sigma$ we find with $\sqrt{\sigma} = 440$ MeV

$$\chi = (240\ \text{MeV})^4. \tag{5.70}$$

This value is somewhat larger than the lattice prediction $\chi = (200\text{–}230\ \text{MeV})^4$.

## 5.5 Summary and conclusions

We have calculated the topological susceptibility $\chi$ within the Hamiltonian approach to Yang–Mills theory in Coulomb gauge using the gluon and ghost propagators determined previously by a variational solution of the Yang–Mills Schrödinger equation as input. To our knowldedge this is the first *ab-initio* continuum calculation of the topological susceptibility, granted the approximations adopted. The precise numerical value for $\chi$ depends on the relation between the Coulomb string tension and the one extracted from the Wilson loop. It is therefore very desirable to perform a sophisticated calculation of the Wilson loop within the present approach. Adopting the relation $\sigma_{\rm C} = 1.5\,\sigma$ as suggested by lattice calculations, the numerical value obtained for $\chi$ is in reasonable agreement with the lattice data. The results obtained in the present paper are quite encouraging for the calculation of further hadronic quantities within the present approach. This will require the inclusion of the quarks.

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## Appendix

In the following, we show how the wave functional Eq. (5.19) can be modified to fulfill Eq. (5.18), so that $\theta$ becomes a true angle variable. For this purpose consider the wave functional

$$\bar{\Psi}_\theta[A] = \int_0^{2\pi} \mathrm{d}\theta' \, \delta[\theta, \theta'] \, \Psi_{\theta'}[A], \tag{5.71}$$

where

$$\delta[\theta, \theta'] = \frac{1}{2\pi} \sum_m \mathrm{e}^{-\mathrm{i}m(\theta-\theta')} \tag{5.72}$$

is the periodic $\delta$-function satisfying $\delta[\theta + 2\pi, \theta'] = \delta[\theta, \theta']$. Inserting Eq. (5.19) into Eq. (5.71) we obtain

$$\bar{\Psi}_\theta[A] = \sum_m \mathrm{e}^{-\mathrm{i}\theta m} f(m - W[A]) \, \phi[A], \tag{5.73}$$

where

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta' \, \mathrm{e}^{\mathrm{i}\theta' x} \, . \tag{5.74}$$

The functional $\bar{\Psi}_\theta[A]$ satisfies both Eq. (5.17) and Eq. (5.18) and thus can be used as the wave functional of the $\theta$-vacuum. According to Eq. (5.73), it has the form

$$\bar{\Psi}_\theta[A] = \sum_m \mathrm{e}^{-\mathrm{i}\theta m} \, \Psi_m[A], \tag{5.75}$$

where

$$\begin{aligned} \Psi_m[A] &= f(m - W[A]) \, \phi[A] \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta' \, \mathrm{e}^{\mathrm{i}m\theta'} \, \Psi_{\theta'}[A] \end{aligned} \tag{5.76}$$

is a localized wave functional centered at the classical vacuum $U\partial U^\dagger$ with winding number $n[U] = m$. It satisfies the relation

$$\Psi_m[A^U] = \Psi_{m-n[U]}[A]. \tag{5.77}$$

Replacing $\bar{\Psi}_\theta[A]$ by $\Psi_m[A]$ corresponds to the so-called 'tight binding' approximation in solid states physics.

We are not using this approximation. From Eq. (5.76) it is seen that our wave functional $\Psi_\theta[A]$ is the Fourier transform of $\Psi_m[A]$.

Using the Poisson relation, the periodic $\delta$-function Eq. (5.72) can be expressed as

$$\delta[\theta,\theta'] = \sum_{l=-\infty}^{\infty} \delta(\theta - \theta' + 2\pi l), \tag{5.78}$$

where $\delta(x)$ denotes the ordinary $\delta$-function. Inserting this representation into Eq. (5.71) we find

$$\bar{\Psi}_\theta[A] = \Psi_{\theta+2\pi l_0}[A], \tag{5.79}$$

where $l_0$ is defined by the condition

$$\theta + 2\pi l_0 \in [0, 2\pi). \tag{5.80}$$

This shows that $\Psi_\theta[A]$, Eq. (5.19), is a correct representation of the wave functional of the $\theta$-vacuum for $\theta \in [0, 2\pi)$.

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Viresque acquirit eundo.

*(Publius Vergilius Maro)*

# Inclusion of dynamical quarks

In this work, only the purely gluonic sector of QCD has been investigated so far. Although pure Yang–Mills theory is fascinating by itself, the evaluation of hadronic observables requires the inclusion of matter fields. A generalization of the non-perturbative methods described so far to dynamical fermion fields is beyond the scope of this thesis; nevertheless, a short overview of the perturbative behaviour of full QCD will be given in this short chapter.

## 6.1 Perturbative QCD vacuum state

From the QCD Lagrangian Eq. (I.21), which defines the theory of interacting quarks and gauge fields, the QCD Hamilton operator in Coulomb gauge

$$H_{\mathrm{QCD}} = \int \mathrm{d}^d x\, \psi^\dagger(\mathbf{x}) \big[ -\mathrm{i}\, \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} - g\, \boldsymbol{\alpha} \cdot \mathbf{A}(\mathbf{x}) + \beta m \big] \psi(\mathbf{x}) + H_{\mathrm{YM}} \tag{6.1}$$

can be derived with the standard methods. In Eq. (6.1), $\alpha^i$ and $\beta$ are the Dirac matrices,[1] and the (transverse) gauge field is a matrix in the fundamental representation of the colour algebra. The Yang–Mills Hamiltonian $H_{\mathrm{YM}}$ is the same as given in Eq. (I.122) except that the external colour charge has to be replaced by the fermion charge

$$\rho^a_{\mathrm{ext}}(\mathbf{x}) \to \psi^{m\dagger}_\alpha(\mathbf{x})\, t^{mn}_a\, \psi^n_\alpha(\mathbf{x}), \tag{6.2}$$

where the $t^{mn}_a$'s are the group generators in the fundamental representation. (For the sake of brevity, the superscript '$F$' identifying the fundamental representation is discarded throughout this chapter.) Furthermore, in this chapter Greek subscripts are Dirac spinor indices. In the quantized theory, the fermion fields satisfy the canonical anticommutation relation

$$\big\{ \psi^m_\alpha(\mathbf{x}), \psi^{n\dagger}_\beta(\mathbf{y}) \big\} = \delta^{mn} \delta_{\alpha\beta}\, \delta(\mathbf{x} - \mathbf{y}). \tag{6.3}$$

The perturbative analysis can be carried out along the same lines of Article 2. The fermion field operator is decomposed in Fourier modes

$$\begin{aligned} \psi^m_\alpha(\mathbf{x}) &= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\, \psi^m_\alpha(\mathbf{p}), \\ \psi^m_\alpha(\mathbf{p}) &= \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \big[ b^m(\mathbf{p}, s) u_\alpha(\mathbf{p}, s) + d^{m\dagger}(-\mathbf{p}, s) v_\alpha(-\mathbf{p}, s) \big], \end{aligned} \tag{6.4}$$

[1] The non-covariant form of the Dirac matrices is used here, with $\beta \equiv \gamma^0$ and $\alpha^i \equiv \gamma^0 \gamma^i$.

where

$$E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m^2} \tag{6.5}$$

is the fermion "energy", and $s$ labels the two independent spinor solutions of the Dirac equation. In order to fulfil the anticommutation relation (6.3), the ladder operators must satisfy the anticommutation rules

$$\left\{b^m(\mathbf{p},s), b^{n\dagger}(\mathbf{q},s')\right\} = \delta^{mn}\delta_{ss'}\delta(\mathbf{p}-\mathbf{q}) = \left\{d^m(\mathbf{p},s), d^{n\dagger}(\mathbf{q},s')\right\}, \tag{6.6}$$

with all other anticommutators vanishing. The Dirac spinors $u$, $v$ appearing in Eq. (6.4) are normalized to

$$u^\dagger(\mathbf{p},s)\,u(\mathbf{p},s') = 2E_{\mathbf{p}}\,\delta_{ss'} = v^\dagger(\mathbf{p},s)\,v(\mathbf{p},s'), \tag{6.7a}$$

$$u^\dagger(\mathbf{p},s)\,v(-\mathbf{p},s') = 0. \tag{6.7b}$$

For later convenience, it is useful to introduce the following orthogonal projectors

$$S^{(\pm)}(\mathbf{p}) := \frac{E_{\mathbf{p}} \pm h_{\mathrm{D}}(\mathbf{p})}{2E_{\mathbf{p}}}\,, \tag{6.8}$$

where

$$h_{\mathrm{D}}(\mathbf{p}) = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m \tag{6.9}$$

is the Dirac operator. These are projectors in the sense that

$$\begin{aligned} S^{(+)}(\mathbf{p})\,u(\mathbf{p},s) &= u(\mathbf{p},s), & S^{(+)}(\mathbf{p})\,v(-\mathbf{p},s) &= 0,\\ S^{(-)}(\mathbf{p})\,v(-\mathbf{p},s) &= v(-\mathbf{p},s), & S^{(-)}(\mathbf{p})\,u(\mathbf{p},s) &= 0, \end{aligned} \tag{6.10}$$

and that they satisfy

$$S^{(+)}(\mathbf{p}) + S^{(-)}(\mathbf{p}) = \mathbb{1}, \qquad S^{(+)}(\mathbf{p})\,S^{(-)}(\mathbf{p}) = 0, \qquad [S^{(\pm)}(\mathbf{p})]^2 = S^{(\pm)}(\mathbf{p}). \tag{6.11}$$

Furthermore, the projectors $S^{(\pm)}$ are related to the Dirac spinors by the sum rules

$$\sum_s \frac{u(\mathbf{p},s)\otimes u^\dagger(\mathbf{p},s)}{2E_{\mathbf{p}}} = S^{(+)}(\mathbf{p}), \qquad \sum_s \frac{v(-\mathbf{p},s)\otimes v^\dagger(-\mathbf{p},s)}{2E_{\mathbf{p}}} = S^{(-)}(\mathbf{p}). \tag{6.12}$$

With the help of the orthogonality relations (6.7), the free fermionic Hamilton operator can be rewritten in terms of the ladder operators as

$$\begin{aligned} H_{\mathrm{D}} &= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\psi^\dagger(\mathbf{p})\,h_{\mathrm{D}}(\mathbf{p})\,\psi(\mathbf{p})\\ &= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,E_{\mathbf{p}}\left[b^{m\dagger}(\mathbf{p},s)\,b^m(\mathbf{p},s) + d^{m\dagger}(\mathbf{p},s)\,d^m(\mathbf{p},s)\right] + E_0\,, \end{aligned} \tag{6.13}$$

where $E_0$ is the (negative divergent) zero-point energy. The vacuum state of the free theory is annihilated by the operators $b$ and $d$,

$$b^m(\mathbf{p},s)\,|0\rangle = 0, \qquad d^m(\mathbf{p},s)\,|0\rangle = 0. \tag{6.14}$$

Their Hermitian conjugate operators generate the eigenstates of $H_{\mathrm{D}}$,

$$H_{\mathrm{D}}\,b^{m\dagger}(\mathbf{p},s)\,|0\rangle = (E_0 + E_{\mathbf{p}})b^{m\dagger}(\mathbf{p},s)\,|0\rangle; \tag{6.15}$$

an analogous relation holds for $d^\dagger$.

In the Hamiltonian approach, the fermion propagator is defined by

$$S^{mn}_{\alpha\beta}(\mathbf{p})\,\delta\!\!\!^{-}(\mathbf{p}-\mathbf{q}) := \left\langle \tfrac{1}{2}\big[\psi^m_\alpha(\mathbf{p}), \psi^{n\dagger}_\beta(\mathbf{q})\big]\right\rangle . \tag{6.16}$$

The commutator occurring in Eq. (6.16) is due to the ambiguity of the equal-time limit in the time ordering. Exploiting the canonical anticommutation relations Eq. (6.3), Eq. (6.16) can be rewritten as

$$S^{mn}_{\alpha\beta}(\mathbf{p})\,\delta\!\!\!^{-}(\mathbf{p}-\mathbf{q}) = \left\langle \psi^m_\alpha(\mathbf{p})\,\psi^{n\dagger}_\beta(\mathbf{q})\right\rangle - \tfrac{1}{2}\,\delta^{mn}\,\delta_{\alpha\beta}\,\delta\!\!\!^{-}(\mathbf{p}-\mathbf{q}). \tag{6.16$'$}$$

Equation (6.16) [(6.16′)] is the quantity which should be compared with the equal-time propagator obtained in the Lagrangian approach by integrating out the temporal component of the four-momentum. The free fermion propagator is given by the expectation value in Eq. (6.16) evaluated in the vacuum state defined by Eq. (6.14). With the help of the anticommutation relations Eq. (6.6) and of the orthogonality properties Eqs. (6.7) of the Dirac spinors, the free propagator can be easily evaluated,

$$S^{(0)}(\mathbf{p}) = \frac{h_{\mathrm{D}}(\mathbf{p})}{2E_{\mathbf{p}}}\,. \tag{6.17}$$

This propagator is related to the projectors $S^{(\pm)}$ [Eq. (6.8)] by

$$S^{(\pm)}(\mathbf{p}) = \frac{1}{2} \pm S^{(0)}(\mathbf{p}), \qquad \frac{1}{2}\big[S^{(+)}(\mathbf{p}) - S^{(-)}(\mathbf{p})\big] = S^{(0)}(\mathbf{p}). \tag{6.18}$$

The perturbative expansion of the Hamilton operator can be performed as described in Sec. 2.2.2. The first-order perturbation stems from the minimal coupling in Eq. (6.1),

$$\begin{aligned} H^{(q)}_1 &= \int \mathrm{đ}^d p_1\ \mathrm{đ}^d p_2\ \psi^\dagger(\mathbf{p}_1)\big[-g\,\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{p}_1-\mathbf{p}_2)\big]\psi(\mathbf{p}_2) \\ &= -g\,t^{mn}_a\,\alpha^i_{\alpha\beta}\int \mathrm{đ}^d p_1\ \mathrm{đ}^d p_2\ \psi^{m\dagger}_\alpha(\mathbf{p}_1)\,A^a_i(\mathbf{p}_1-\mathbf{p}_2)\,\psi^n_\beta(\mathbf{p}_2), \end{aligned} \tag{6.19a}$$

while the second order perturbative operator arises from the Coulomb interaction

$$\begin{aligned} H^{(q)}_2 = t^{mn}_a\,t^{kl}_a\,\frac{1}{2}\int \mathrm{đ}^d[p_1p_2p_3p_4]\,\delta\!\!\!^{-}(\mathbf{p}_1-\mathbf{p}_2+\mathbf{p}_3-\mathbf{p}_4) \\ \times\,\psi^{m\dagger}_\alpha(\mathbf{p}_1)\,\psi^n_\alpha(\mathbf{p}_2)\,\frac{1}{(\mathbf{p}_1-\mathbf{p}_2)^2}\,\psi^{k\dagger}_\beta(\mathbf{p}_3)\,\psi^l_\beta(\mathbf{p}_4). \end{aligned} \tag{6.19b}$$

Equation (6.19b) represents the Coulomb interaction of the quark charge densities. From the expansion of the Coulomb term in the Yang–Mills Hamiltonian Eq. (I.122) also a mixed term arises, which represents the interaction between the fermion and the gauge field charge [see Eq. (2.81)]. This term, however, does not matter for the one-loop evaluation of the propagators, and will henceforth be discarded.

The corrections to the QCD vacuum wave functional can be evaluated by means of the fermionic generalizations of Eqs. (2.38). The vacuum functional is, at lowest order, the tensor product of the vacuum functionals of each sector,

$$|0\rangle_{\mathrm{QCD}} = |0\rangle_{\mathrm{YM}} \otimes |0\rangle_{\mathrm{Q}}, \tag{6.20}$$

and vacuum expectation values can be factorized in products of fermionic and gluonic terms. In the following, the tensor product will be not explicitly written, and it is implied that fermion operators act on $|0\rangle_{\mathrm{Q}}$, while gauge field operators act on $|0\rangle_{\mathrm{YM}}$. Furthermore, with the help of Eq. (6.10), the closure relation in the free theory can be rewritten in terms of field operators as

$$\begin{aligned}\sum_{m,s,\mathbf{p}} b^{m\dagger}(\mathbf{p},s)\,|0\rangle\langle 0|\,b^{m}(\mathbf{p},s) &= \sum_{m,\mathbf{p}} S^{(+)}_{\alpha\beta}(\mathbf{p})\,\psi^{m\dagger}_{\alpha}(\mathbf{p})\,|0\rangle\langle 0|\,\psi^{m}_{\beta}(\mathbf{p}),\\ \sum_{m,s,\mathbf{p}} d^{m\dagger}(\mathbf{p},s)\,|0\rangle\langle 0|\,d^{m}(\mathbf{p},s) &= \sum_{m,\mathbf{p}} S^{(-)}_{\alpha\beta}(\mathbf{p})\,\psi^{m}_{\beta}(\mathbf{p})\,|0\rangle\langle 0|\,\psi^{m\dagger}_{\alpha}(\mathbf{p}).\end{aligned} \tag{6.21}$$

This allows the matrix elements occurring in Eqs. (2.38) to be expressed in terms of field operators only, for which Wick's theorem holds. Equation (6.21) is similar to the representation Eq. (2.37) used for the gluonic states. However, whilst Eq. (2.37) is a simplification which needs a supplementary calculational rule, Eq. (6.21) is an exact identity.

Using Eq. (2.38a), the first-order correction

$$|0\rangle^{(1)} = \sum t_a^{mn}\,\frac{\left[S^{(+)}(\mathbf{p})\,\alpha^i\,S^{(-)}(-\mathbf{q})\right]_{\alpha\beta}}{E_{\mathbf{p}}+E_{\mathbf{q}}+|\mathbf{p}+\mathbf{q}|}\,\psi^{m\dagger}_{\alpha}(\mathbf{p})\,\psi^{n}_{\beta}(-\mathbf{q})\,A^a_i(\mathbf{p}+\mathbf{q})\,|0\rangle \tag{6.22}$$

arises from the first-order perturbative operator Eq. (6.19a), the quark-gluon vertex. The second-order perturbative contribution to the vacuum wave functional contains a fermionic part

$$\begin{aligned}|0\rangle^{(2)} = C_F \sum\Bigg[S^{(+)}(\mathbf{p})\left(\frac{t_{ij}(\mathbf{p}+\mathbf{q})}{2|\mathbf{p}+\mathbf{q}|}\,\frac{\alpha^i S^{(0)}(-\mathbf{q})\alpha^j}{E_{\mathbf{p}}+E_{\mathbf{q}}+|\mathbf{p}+\mathbf{q}|}-\frac{S^{(0)}(-\mathbf{q})}{2(\mathbf{p}+\mathbf{q})^2}\right)S^{(-)}(\mathbf{p})\Bigg]_{\alpha\beta}\\ \times\frac{1}{E_{\mathbf{p}}}\,\psi^{m\dagger}_{\alpha}(\mathbf{p})\psi^{m}_{\beta}(\mathbf{p})\,|0\rangle,\end{aligned} \tag{6.23}$$

as well as a term involving gauge field operators

$$|0\rangle^{(2)} = \frac{1}{4}\sum\frac{\mathrm{tr}\left[S^{(+)}(\mathbf{p})\,\alpha^i\,S^{(-)}(-\mathbf{q})\,\alpha^j\right]}{E_{\mathbf{p}}+E_{\mathbf{q}}+|\mathbf{p}+\mathbf{q}|}\,\frac{1}{|\mathbf{p}+\mathbf{q}|}\,A^a_i(-\mathbf{p}-\mathbf{q})\,A^a_j(\mathbf{p}+\mathbf{q})\,|0\rangle. \tag{6.24}$$

In Eq. (6.23), $C_F$ is the Casimir invariant of the fundamental representation of $SU(N_{\mathrm{c}})$

$$C_F = \frac{N_{\mathrm{c}}^2-1}{2N_{\mathrm{c}}}\,. \tag{6.25}$$

The fermionic second-order term Eq. (6.23) contributes only to the quark propagator, while the gauge field contribution Eq. (6.24) yields a quark-loop term to the gluon propagator. On the other hand, the first-order term Eq. (6.22) gives a one-loop correction to both the gluon and the quark propagator.

Equations (6.22) and (6.23) can be rewritten also in terms of the ladder operators by means of Eqs. (6.4) and (6.10), yielding

$$|0\rangle^{(1)} = \sum\frac{u^\dagger(\mathbf{p},s)\,\alpha^i\,v(\mathbf{q},s')}{E_{\mathbf{p}}+E_{\mathbf{q}}+|\mathbf{p}+\mathbf{q}|}\,\frac{t_a^{mn}}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\,b^{m\dagger}(\mathbf{p},s)\,d^{n\dagger}(\mathbf{q},s')\,A^a_i(\mathbf{p}+\mathbf{q})\,|0\rangle \tag{6.22$'$}$$

for the first-order correction, and

$$|0\rangle^{(2)} = C_F \sum u^\dagger(\mathbf{p},s)\left(\frac{t_{ij}(\mathbf{p}+\mathbf{q})}{2|\mathbf{p}+\mathbf{q}|}\frac{\alpha^i S^{(0)}(-\mathbf{q})\alpha^j}{E_\mathbf{p}+E_\mathbf{q}+|\mathbf{p}+\mathbf{q}|} - \frac{S^{(0)}(-\mathbf{q})}{2(\mathbf{p}+\mathbf{q})^2}\right)v(-\mathbf{p},s') \times \frac{1}{2E_\mathbf{p}^2}\, b^{m\dagger}(\mathbf{p},s)d^{m\dagger}(-\mathbf{p},s')\,|0\rangle \quad (6.23')$$

for the fermionic second-order term.

## 6.2 Gluon propagator and $\beta$ function

The correction to the gluon propagator can be evaluated as described in Sec. 2.4.1. The evaluation of the matrix elements is straightforward, and the dynamical quark fields yield an additional loop integral to the one-loop form factor given in Eq. (2.62)

$$D_A^{(1)}(\mathbf{p})_q = \frac{t_{ij}(\mathbf{p})}{2(d-1)\mathbf{p}^2}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{2|\mathbf{p}|+E_\mathbf{q}+E_{\mathbf{p}+\mathbf{q}}}{(|\mathbf{p}|+E_\mathbf{q}+E_{\mathbf{p}+\mathbf{q}})^2}\;\mathrm{tr}\big[S^{(+)}(\mathbf{p}+\mathbf{q})\,\alpha^i\,S^{(-)}(\mathbf{q})\,\alpha^j\big]. \quad (6.26)$$

The Dirac trace can be explicitly worked out with the usual rules, and Eq. (6.26) can be rewritten as

$$D_A^{(1)}(\mathbf{p})_q = \frac{2}{(d-1)\mathbf{p}^2}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{2|\mathbf{p}|+E_\mathbf{q}+E_{\mathbf{p}+\mathbf{q}}}{(|\mathbf{p}|+E_\mathbf{q}+E_{\mathbf{p}+\mathbf{q}})^2} \times \frac{(d-1)\big[E_\mathbf{q}E_{\mathbf{p}+\mathbf{q}}+\mathbf{q}\cdot(\mathbf{p}+\mathbf{q})+m^2\big]-2q_i t_{ij}(\mathbf{p})q_j}{E_\mathbf{q}E_{\mathbf{p}+\mathbf{q}}}. \quad (6.26')$$

As discussed in Articles 2 and 3, the result Eq. (6.26′) can be compared with the result from the Lagrangian approach. The quark-loop contribution to the gluon form factor evaluated in Ref. [1] reads

$$D_{AA}^{(1)}(p)_q = \frac{2}{(d-1)p^2}\int\frac{\mathrm{d}^{d+1} q}{(2\pi)^{d+1}}\,\frac{(d-1)\big[q_4^2+q_4p_4+\mathbf{q}\cdot(\mathbf{q}+\mathbf{p})+m^2\big]-2q_i t_{ij}(\mathbf{p})q_j}{(q^2+m^2)[(p+q)^2+m^2]}, \quad (6.27)$$

where $p^2 = p_4^2+\mathbf{p}^2$ is the squared Euclidean four-momentum. The dressing function of the equal-time propagator is related to the dressing function Eq. (6.27) of the energy-dependent propagator by

$$D_{AA}^{(1)}(\mathbf{p})_q = 2|\mathbf{p}|\int\frac{\mathrm{d}p_4}{2\pi}\frac{D_{AA}^{(1)}(p)_q}{p^2}. \quad (6.28)$$

Inserting Eq. (6.27) into Eq. (6.28) and performing the integrals over $p_4$ and $q_4$, Eq. (6.26′) is recovered. Using the results of Ref. [1], the expression Eq. (2.74) for the dimensionally regularized gluon propagator can be rewritten as

$$D_A(\mathbf{p}) = 1+\frac{g^2}{(4\pi)^{2-\varepsilon}}\left[\left(N_\mathrm{c}-\frac{2N_\mathrm{f}}{3}\right)\left(\frac{1}{\varepsilon}-\gamma_\mathrm{E}-\ln\frac{\mathbf{p}^2}{\mu^2}\right)+\cdots\right] \quad (6.29)$$

where it has been assumed that $N_\mathrm{f}$ fermion species are present. From the non-renormalization of the ghost-gluon vertex, the first coefficient of the $\beta$-function becomes

$$\beta_0 = \frac{11N_\mathrm{c}-2N_\mathrm{f}}{3}, \quad (6.30)$$

which is the well-known, gauge-invariant result.

## 6.3 Quark propagator

Starting from Eq. (6.16), the perturbative expansion of the quark propagator can be written as

$$S^{mn}_{\alpha\beta}(\mathbf{p})\delta\!\!\!^{-}(\mathbf{p}-\mathbf{q}) = S^{(0)mn}_{\alpha\beta}(\mathbf{p})\delta\!\!\!^{-}(\mathbf{p}-\mathbf{q}) + \sum_{k+l>0} g^{k+l\,(k)}\langle\!\langle 0|\,\psi^m_\alpha(\mathbf{p})\,\psi^{n\dagger}_\beta(\mathbf{q})\,|0\rangle\!\rangle^{(l)}, \tag{6.31}$$

where $S^{(0)}$ is the perturbative propagator given in Eq. (6.17). The evaluation of the matrix elements arising from the insertion of Eqs. (6.22) and (6.23) into Eq. (6.31) is, although lengthy, straightforward, and the result reads

$$\begin{aligned} S(\mathbf{p}) = S^{(0)}(\mathbf{p}) &\left[1 - g^2 C_F \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(E_\mathbf{p}+E_\mathbf{q}+|\mathbf{p}+\mathbf{q}|)^2} \frac{d-1}{2|\mathbf{p}+\mathbf{q}|}\right] \\ &+ g^2 \frac{C_F}{E_\mathbf{p}} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{2(\mathbf{p}+\mathbf{q})^2} \left[\frac{1}{2}\, S^{(0)}(-\mathbf{q}) - 2S^{(0)}(\mathbf{p})\, S^{(0)}(-\mathbf{q})\, S^{(0)}(\mathbf{p})\right] \\ - g^2 \frac{C_F}{E_\mathbf{p}} &\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{(E_\mathbf{p}+E_\mathbf{q}+|\mathbf{p}+\mathbf{q}|)^2} \frac{t_{ij}(\mathbf{p}+\mathbf{q})}{2|\mathbf{p}+\mathbf{q}|} \Big[(E_\mathbf{q}+|\mathbf{p}+\mathbf{q}|)\frac{1}{2}\alpha^i\, S^{(0)}(-\mathbf{q})\,\alpha^j \\ &- 2(2E_\mathbf{p}+E_\mathbf{q}+|\mathbf{p}+\mathbf{q}|)S^{(0)}(\mathbf{p})\,\alpha^i\, S^{(0)}(-\mathbf{q})\,\alpha^j\, S^{(0)}(\mathbf{p})\Big]. \end{aligned} \tag{6.32}$$

There are now several ways to define form factors for the static propagator. For example, a possible choice is to define form factors for the inverse propagator,

$$[S^{(0)}(\mathbf{p})]^{-1} = 2\,\frac{\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m}{\sqrt{\mathbf{p}^2+m^2}} \quad\Rightarrow\quad S(\mathbf{p})^{-1} = 2A(\mathbf{p})\,\frac{\boldsymbol{\alpha}\cdot\mathbf{p}+\beta M(\mathbf{p})}{\sqrt{\mathbf{p}^2+M^2(\mathbf{p})}}\,. \tag{6.33}$$

Although this would be the most appropriate choice in a non-perturbative approach, a different parametrization for the propagator will be assumed here

$$S(\mathbf{p}) = \frac{\mathcal{A}(\mathbf{p})\,\boldsymbol{\alpha}\cdot\mathbf{p}+\beta\,\mathcal{B}(\mathbf{p})}{2E_\mathbf{p}}. \tag{6.34}$$

This form is more convenient to investigate the renormalization of the propagator at one-loop order and to compare it with the result from the Lagrangian approach [1]. The one-loop expressions for the form factors $\mathcal{A}$, $\mathcal{B}$ can be extracted from Eq. (6.32) by

$$\mathcal{A}(\mathbf{p}) = \frac{E_\mathbf{p}\,p_i}{2\mathbf{p}^2}\;\mathrm{tr}\big[\alpha^i S(\mathbf{p})\big], \qquad \mathcal{B}(\mathbf{p}) = \frac{E_\mathbf{p}}{2}\;\mathrm{tr}\big[\beta S(\mathbf{p})\big]. \tag{6.35}$$

The evaluation of the traces is straightforward, and results in

$$\begin{aligned} \mathcal{A}(\mathbf{p}) = 1 &- g^2 \frac{m^2 C_F}{2\mathbf{p}^2 E_\mathbf{p}^2} \int \frac{\mathrm{d}^d q}{(2\pi)^d}\, \frac{\mathbf{p}\cdot(\mathbf{p}+\mathbf{q})}{E_\mathbf{q}(\mathbf{p}+\mathbf{q})^2} \\ &- g^2 \frac{C_F}{2\mathbf{p}^2 E_\mathbf{p}^2} \int \frac{\mathrm{d}^d q}{(2\pi)^d}\, \frac{1}{E_\mathbf{q}|\mathbf{p}+\mathbf{q}|(E_\mathbf{p}+E_\mathbf{q}+|\mathbf{p}+\mathbf{q}|)^2} \\ \times \Bigg\{ &\left[(d-3)\mathbf{p}\cdot\mathbf{q} + 2\frac{[\mathbf{p}\cdot(\mathbf{p}+\mathbf{q})][\mathbf{q}\cdot(\mathbf{p}+\mathbf{q})]}{(\mathbf{p}+\mathbf{q})^2}\right]\big[m^2(E_\mathbf{p}+E_\mathbf{q}+|\mathbf{p}+\mathbf{q}|) - \mathbf{p}^2 E_\mathbf{p}\big] \\ &+ (d-1)\mathbf{p}^2\big[E_\mathbf{p}^2 E_\mathbf{q} + m^2(2E_\mathbf{p}+E_\mathbf{q}+|\mathbf{p}+\mathbf{q}|)\big]\Bigg\} \end{aligned} \tag{6.36a}$$

for the form factor of the kinetic term, and

$$\mathcal{B}(\mathbf{p}) = m + m\,g^2 \frac{C_F}{2E_{\mathbf{p}}^2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathbf{p}\cdot(\mathbf{p}+\mathbf{q})}{E_{\mathbf{q}}(\mathbf{p}+\mathbf{q})^2} + m\,g^2 \frac{C_F}{2E_{\mathbf{p}}^2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{E_{\mathbf{q}}|\mathbf{p}+\mathbf{q}|(E_{\mathbf{p}}+E_{\mathbf{q}}+|\mathbf{p}+\mathbf{q}|)^2} \times \left\{ (2E_{\mathbf{p}}+E_{\mathbf{q}}+|\mathbf{p}+\mathbf{q}|) \left[ (d-3)\mathbf{p}\cdot\mathbf{q} + 2\frac{[\mathbf{p}\cdot(\mathbf{p}+\mathbf{q})][\mathbf{q}\cdot(\mathbf{p}+\mathbf{q})]}{(\mathbf{p}+\mathbf{q})^2} \right] + (d-1)\big[(\mathbf{p}^2-m^2)E_{\mathbf{p}} + \mathbf{p}^2|\mathbf{p}+\mathbf{q}| - m^2 E_{\mathbf{q}}\big] \right\} \quad (6.36b)$$

for the mass term.

As discussed in Articles 2 and 3, these form factors can be compared with the results of the Lagrangian approach. In Ref. [1], the quark propagator (in Euclidean space) is parametrized as

$$S(p) = \frac{p_4 F_t(p) + \boldsymbol{\alpha}\cdot\mathbf{p} F_s(p) + \beta M(p)}{p_4^2 + \mathbf{p}^2 + m^2} \quad (6.37)$$

where all dressing functions are functions of both $\mathbf{p}^2$ and $p_4^2$. In Eq. (6.37) the component of the form $p_4 p_i$ has been discarded, since it does not arise at one-loop level.[2]

The static propagator is obtained by the energy-dependent one [Eq. (6.37)] by integrating out the temporal component $p_4$ of the four-momentum. Since the dressing function $F_t(p)$ is an even function of $p_4$, this component vanishes in the equal-time limit. After performing also the integral over the temporal component of the loop momentum, the other two dressing functions reduce to Eqs. (6.36a) and (6.36b),

$$\int \frac{\mathrm{d}p_4}{2\pi} \frac{F_s(p)}{p_4^2+\mathbf{p}^2+m^2} = \mathcal{A}(\mathbf{p}), \qquad \int \frac{\mathrm{d}p_4}{2\pi} \frac{M(p)}{p_4^2+\mathbf{p}^2+m^2} = \mathcal{B}(\mathbf{p}). \quad (6.38)$$

The renormalization of the quark propagator presents no significant difficulty. Performing the integral in some regularization scheme, the form factors have the form

$$\mathcal{A}(\mathbf{p}) = 1 + g^2 a_1, \qquad \mathcal{B}(\mathbf{p}) = m(1 + g^2 b_1), \quad (6.39)$$

with some coefficients $a_1$, $b_1$. Inserting this into Eq. (6.34) yields

$$S(\mathbf{p}) = \frac{1}{2\sqrt{\mathbf{p}^2+m^2}} \left[ \boldsymbol{\alpha}\cdot\mathbf{p}(1+g^2 a_1) + \beta m(1+g^2 b_1) \right]. \quad (6.40)$$

The renormalized mass $m_R$ can be introduced as usual by

$$m = Z_m m_R, \qquad Z_m = 1 + g^2 c_1, \quad (6.41)$$

which allows Eq. (6.40) to be rewritten as

$$S(\mathbf{p}) = \left( 1 - g^2 \frac{c_1 m_R^2}{\mathbf{p}^2+m_R^2} \right) \frac{\boldsymbol{\alpha}\cdot\mathbf{p}(1+g^2 a_1) + \beta m_R[1+g^2(b_1+c_1)]}{2\sqrt{\mathbf{p}^2+m_R^2}}. \quad (6.42)$$

[2] Lattice calculations [2] indicate that this component vanishes.

The term in the parentheses in Eq. (6.42) arises from replacing the bare mass $m$ in the denominator of Eq. (6.40) by the renormalized mass $m_R$. Since the bare propagator is related to the renormalized one by

$$S(\mathbf{p}) = Z_2 \, \frac{\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m_R}{2\sqrt{\mathbf{p}^2 + m_R^2}} \,, \tag{6.43}$$

the mass and quark wave function renormalization constants $Z_m$ and $Z_2$ must be defined by

$$Z_m = 1 + g^2(a_1 - b_1), \qquad Z_2 = 1 + g^2 \frac{\mathbf{p}^2 a_1 + m_R^2 b_1}{\mathbf{p}^2 + m_R^2}. \tag{6.44}$$

The coefficients $a_1$, $b_1$ can be obtained either by integrating the results of the Lagrangian approach over the temporal component $p_4$ of the four-momentum, or by evaluating the integrals Eqs. (6.36) with a cut-off momentum $\Lambda$. In the first case, the divergent parts are with $d = 3 - 2\varepsilon$

$$a_1 = -\frac{C_F}{(4\pi)^2} \, \frac{\mathbf{p}^2 + 4m_R^2}{\mathbf{p}^2 + m_R^2} \, \frac{1}{\varepsilon} \,, \qquad b_1 = \frac{C_F}{(4\pi)^2} \, \frac{2\mathbf{p}^2 - m_R^2}{\mathbf{p}^2 + m_R^2} \, \frac{1}{\varepsilon} \,. \tag{6.45}$$

The coefficients in front of the $\varepsilon$ pole in Eq. (6.45) are found as factors multiplying $\ln \Lambda^2$ when evaluating the integrals with a cut-off momentum $\Lambda$. The two form factors $\mathcal{A}$, $\mathcal{B}$ cannot be separately renormalized, since the coefficients in Eq. (6.45) are momentum dependent. However, the quark mass and wave function renormalization constants [see Eq. (6.44)] result in the momentum-independent quantities

$$Z_m = 1 - \frac{g^2 C_F}{(4\pi)^2} \, \frac{3}{\varepsilon} + \cdots \,, \tag{6.46a}$$

$$Z_2 = 1 - \frac{g^2 C_F}{(4\pi)^2} \, \frac{1}{\varepsilon} + \cdots \,, \tag{6.46b}$$

where the renormalization scheme dependent terms have been discarded. In particular, Eq. (6.46a) is the (at this order perturbatively) gauge invariant result for the mass renormalization constant, which agrees with the results for the pole mass from covariant gauges (see e.g. Refs. [3–5]).

It should be remarked here that this renormalization program shows the necessity of working with the propagator defined by Eq. (6.16). If the commutator is not included in the definition, the resulting propagator is not multiplicatively renormalizable.

## References

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I love it
when a plan comes together.

*(John 'Hannibal' Smith)*

# Conclusions

In this work, the ground state of pure Yang–Mills theories has been investigated within the Hamiltonian approach in Coulomb gauge. A thorough one-loop perturbative analysis of the propagators has been performed. Because of asymptotic freedom, the full vacuum wave functional is expected to exhibit the same behaviour at high energies as the perturbative ground state. The numerical calculations performed so far in a non-perturbative variational approach did confirm this expectation. However, it is interesting to notice that the anomalous dimensions of the propagators extracted both from those calculations and from the lattice differ significantly from the corresponding one-loop results. While it is known that the two-loop corrections to the anomalous dimensions in Landau gauge are small compared with the leading order, the Coulomb gauge state of the art lacks such knowledge. In fact, no two-loop results for the Yang–Mills propagators and vertices in Coulomb gauge have been obtained yet, either in the Lagrangian or in the Hamiltonian approach. The only two-loop calculation [Nucl. Phys. **B575**, 359 (2000)] focused on the cancellation of the energy divergences. It would be conceivable that such corrections are comparable to the leading order results. Moreover, it should be recalled that, in the Hamiltonian approach, the number of perturbative operators grows with the perturbative order. It should also be reminded that the renormalization group improvements which led to the anomalous dimensions Eq. (3.97) [see p. 76] rested on the assumption of multiplicative renormalizability, which has not yet been proven for Coulomb gauge. The discrepancy between the one-loop expressions for the anomalous dimensions and the non-perturbative results is thus hardly a surprise.

A general method to treat non-Gaussian wave functionals has also been presented. The procedure relies on well-developed Dyson–Schwinger techniques, and one can draw from the experience gathered on the subject over the last years. Nevertheless, several issues still require some light to be shed on them. As in all continuum approaches, a truncation scheme to handle the infinite tower of coupled integral equations is necessary. In the case of the variational approach, one is confronted with two systems of equations—the Dyson–Schwinger equations connecting the proper vertex functions with the kernels occurring in the ansatz for the wave functional, and the variational equations stemming from the minimization of the vacuum expectation value of the Hamilton operator. At the order considered in Article 4, no ambiguity arises, at least for the gluon propagator. However, when dealing with the higher-order kernels, the interplay between the two systems of equations, and how the truncation procedures interfere with each other, has to be carefully investigated. In pursuing this line of research, this is possibly one of the most fundamental issues of the variational approach in the Hamiltonian framework. While the higher-order kernels do not influence the infrared sector of the theory, they give significant contributions to the mid-momentum regime, on which hadronic physics crucially depends.

An improvement of the wave functional in the GeV region is thus necessary.

While the evaluated vacuum wave functional seems to grasp the essential features of the infrared behaviour of Yang–Mills theories, its final use (and one of particle physics' main goals at all) is to calculate hadronic observables. A first step in this direction has been taken in Article 5. Although some of the approximations involved there are rather crude, the result is quite encouraging. The reason for investigating the topological susceptibility is that it can be evaluated in the pure gauge sector. Further developments require of course the non-perturbative inclusion of dynamical quarks.

Ἐν γὰρ τοῖς ἔργοις αὐτοῦ ἀναστρεφόμενοι διερευνῶσιν καὶ πείθονται τῇ ὄψει, ὅτι καλὰ τὰ βλεπόμενα.

*'For they search busily among his works, but are distracted by what they see, because the things seen are fair.' (Wis. 13,7)*